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Arithmetical Algebra in the Islamic History of Mathematics and Its Peak in the 9th/15th Century: Ibn al-Hāʾim's *al-Mumtiʿ*

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Abstract: Algebra, defined as a method to determine the unknown by means of what is known, given the link between the two, took its initial steps toward disciplinary status during the third/ninth century when al-Khwārizmī produced the first systematic study on the subject. Later Muslim mathematicians followed his lead due to this novel discipline's propensity for improvement and beneficial application. Thus they applied arithmetic to algebra to make it more practical and open and, as a result, derived great benefits from employing it in matters of inheritance, commerce, land surveys, architecture, and other areas. Roughly 550 years after its formation as a discipline, algebra reached its peak in the aforementioned areas. One of its most famous practitioners, Ibn al-Hā'im, had a lasting and widespread influence first with his commentary on Yāsamīnī and then with his versified work al-Muqniʿ and its commentary al-Mumtiʿ. However, the latter work eluded the researchers' attention – perhaps it was overshadowed by the former or lost among the other commentaries – despite its remarkable presentation of the entire conceptual and methodical repertoire of algebra as it was known at that time, not to mention its analysis of the problems and discussion of the philosophical implications in a long-lasting debate on Islamic mathematical history: Should algebra be arithmetical, geometrical, or both? Which track would be more conducive to improving the discipline so it could break new ground in the historical studies of mathematics? Thus, this article seeks to present the status of Ibn al-Hāim's al-Mumtiʿ fī sharh al-Muqniʿ in the history of mathematics, along with its outstanding features and mathematical analysis.

Keywords: mathematics, algebra, Ibn Haim, al-Muqni, al-mumti.

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As a human edifice built on reality, knowledge is a byproduct of processing the data acquired through the senses and observation by mathematical and conceptual structures. In other words, it is the process of acquiring knowledge of the unknown from what is known (i.e., the data at hand), given the link between the two, by various means, methods, and instruments. Therefore, the correspondence of the knowledge elicited to truth is due first to the truth of the data derived via sensation and observation and then its proper processing by the appropriate mathematical or conceptual model. Many questions arise at this juncture, such as "Is it possible to verify data based on sensation and observation? If not, how reliable are they? Is it possible for them to bring out true knowledge once they are run through mental processes? What are the mental structures that provide true knowledge?" In the intellectual history of Islamic civilization, the main stance was to improve the certainty of the primary and secondary steps leading to the formation of knowledge in order to resolve the aforementioned problems. The investments made in terms of instruments of observation from this civilization's earliest days to improve certainty in logic and mathematics as two basic methods in data processing can be taken as evidence of this. Until the seventh/thirteenth century, logic was commonly thought to be the language of the universe and the necessary means to its interpretation.[1] From that time onward, mathematical structures were also thought to be beneficial for the same purpose, and thus the view of attending to different perspectives was adopted.[2] This shift/transformation reinforced the improvement and ongoing pursuit of certainty in all branches of mathematics.

A good example of this in terms of algebra is Ibn al-Hā'im al-Miṣrī's (752/1352-815/1412) *al-Mumtiʿ fī sharḥ al-Muqniʿ*, composed at the beginning of ninth/fifteenth century as a commentary on his own versified work *al-Muqniʿ fī al-jabr wa-al-muqābala*. This synthesis of almost the entire corpus of arithmetical algebra up to his day may be the result of his upbringing in Egypt, located between the eastern and the western halves of the Islamic world (*al-Mashriq* and *al-Maghrib*). In fact, it can be regarded as his era's "peak of arithmetical algebra"[3] due to its fortify-

1 The prevalence of this persuasion, mostly among those involved in natural philosophy and metaphysics, does not change the fact that the mathematical sciences underwent rapid growth since the very beginning of the Islamic civilization.

2 İhsan Fazlıoğlu, "Faal Akıl Ölünce!: XIII Yüzyıl Felsefe-Bilim Tarihi'nde Hakiki (Invisible) ile Zahiri (Visible) İlişkisinin Yeniden Yorumlanması," in *Uluslararası XIII. yüzyılda Felsefe Sempozyumu Bildirileri* (Ankara: Yıldırım Beyazıt Üniversitesi, 2014), 27–36; idem, "Hakikat ile İtibar: Dış-dünya'nın Bilgisinin Doğası Üzerine – XV. Yüzyıl Doğa Felsefesi ve Matematik Açısından Bir İnceleme," *Nazariyat* 1, no. 1 (2014): 1–33.

3 The qualification of "peak" used for the work herein seeks to convey the work's being the most compre-

ing algebraic proofs. This qualification also serves as the reason for writing this article. Moreover, the course of studies in mathematical history of both the classical and Ottoman periods will certainly be changed by considering a work that (1) surveys the field's reserve of knowledge to such a point that it merits the designation of the "encyclopedia of algebra" and (2) incorporating philosophical discussions of the limits of algebra and a germane potential for its growth and expansion.

Before moving on to the author and his work, one should recall the nature of arithmetical algebra and the historical course of its emergence and development, for the path to the peak comes first in the way to the peak, as well as its interpretation and assessment.

I. The Historical Background Of Arithmetical Algebra

Algebra gradually developed two main branches based on which of the two main mathematical values, "shape" and "number," was regarded as the main element and employed to the greatest extent to resolve the problem/equation. The first approach, the building block of geometry and a centerpiece of the algebraic solution, became known as "geometrical algebra," whereas the second approach, which used "number" as the main element, became known as "analytical/arithmetical/numerical algebra." The crux of the differentiation and opposition between the two approaches is the problem of the certain and sound proof for the solution of an algebraic equation. While geometrical algebraists supposed that numerical methods would not always provide sound proof and thus might be misleading, they considered crosschecking with geometrical proof, the one and only method that can provide sound and certain proof, was definitely necessary, even if the solution was obtained via arithmetical methods. Arithmetical algebraists countered by stating that certain proof and crosschecking could also be obtained by numerical methods and that the discipline's improvement, expansion, and applications would be constrained if it relied only on geometrical proof and thus could not move beyond cubical equations. To a large extent history proved them correct, and the discipline continued to improve in the numerical area, especially during the late fifteenth and early sixteenth centuries. However, the attempts of geometrical algebraists to integrate algebra and geometry also bore fruit in the form of Cartesian geom-

hensive, independent title of algebra of its time, one that impeccably utilizes and weaves together the entire body of the field's knowledge, rather than a peak in the form of denoting lower grounds before and after the work.

etry (the geometry of coordinates, now known as "analytical geometry"),[4] which emerged in the seventeenth century after René Descartes (1596–1650), reworked the acquired knowledge in a systematic and theoretical manner in his *La Géométrie*. Consequently, the convenient use of "numbers" both in calculation and oral and written communication (prosaic and symbolic transmission) caught the attention of the Muslim world's mathematicians early on, and the arithmetization of algebra lasted, accelerated, and expanded for 500 years after the fourth/tenth century. Algebra was transmitted to Europe primarily in this arithmetical form.

Considering the earlier manifestations of arithmetical algebra founded by al-Khwārizmī's successors, and continued to develop significantly in the Ottoman classical period, one could trace its manifestation as far back as two thousand years before the starting point of the discipline's first examples.

It would be profitable to deal with the subject under two headings, taking al-Khwārizmī (780–850), the founding figure of algebra, as a starting point, in order to better appreciate the course of development: the algebraic constitution before al-Khwārizmī and its arithmetical character, and the foundation and development of algebra from him to Ibn al-Hā'im.

1. The Primary Examples Of Arithmetical Algebra

In the context of the primary civilizations, the lore of which partly survived the test of time, the idea to arrive at the unknown by means of what is known can be traced to 2000 BCE. While the first salient instances of this idea were attested to in Mesopotamian mathematics, the neighboring Egyptian civilization made headway mostly in geometry. One can detect the idea of algebra in between the lines of geometry and the theory of numbers in Ancient Greece and Alexandria under the heavy influence of Mesopotamian and Egyptian mathematics. Turning to another primary civilization, history also records that the innovations in calculation based especially on the Indian numerals, which include zero and the decimal system, made a significant contribution to algebra in general and to the notion of arithmetical algebra in particular.

4 The foundation of analytical geometry is commonly attributed to René Descartes (1596–1650); however, the first expert primarily on analytical geometry is Julius Plücker (1801-68). For further details, cf. Carl B. Boyer, *A History of Mathematics* (New York: John Wiley and Sons, 1976), 540. For the translation into Turkish, cf. idem, *Matematiğin Tarihi*, trans. Saadet Bağçacı (Istanbul: Doruk Yayınları, 2015), 582–83.

The input from the most recent archaeological excavations indicates that the discipline's seeds can be located in Mesopotamia, particularly in Babylon. The same evidence shows that Mesopotamia's mathematicians were most apt at second-order equations and their solutions in algebra. They organized and studied these equations in nine types and provided a solution for each. The most common two types of these algebraic equations are as follows:

$$\textbf{1.}\ x + y = b \ \textit{\textbf{and}}\ xy = c \longrightarrow x = \frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 - c} \ \textit{\textbf{and}}\ y = \frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 - c}$$

$$\textbf{2.}\ x - y = b \ \textit{\textbf{and}}\ xy = c \longrightarrow x = \sqrt{\left(\frac{b}{2}\right)^2 + c} + \frac{b}{2} \ \textit{\textbf{and}}\ y = \sqrt{\left(\frac{b}{2}\right)^2 + c} - \frac{b}{2}$$

The other kinds of equations were converted into these two types and then solved, for the greater part, in Mesopotamian algebra. In addition, the solutions for the cubical equation ($x^3 + x^2 = 252$) and the equation system ($xy + x - y = 183$, $x + y = 27$) were obtained.[5]

The general pattern of solution methods in Mesopotamian algebra uses geometrical terms and shapes, although it follows the arithmetical method. Employing geometrical terms in the texts should not mislead one, however, for their style of reasoning is fundamentally numerical. Even though the unknown numbers figured as segmented lines and areas, they remained numbers in the mathematicians' minds. Moreover, their main objective was to render calculations, not to provide illustrations or geometrical proofs, even in those problems that appear to be geometrical. Subsequently, an analytical essence behind a geometrical façade is readily manifest in Mesopotamia in general and in Babylonian algebra in particular.[6] To sum up, one might suppose that Mesopotamian mathematicians provided the primitive forms of the arithmetical methods used in algebra. In Egypt, however, the subjects of number theory and geometry predominate. Hence we might move on to the evidence concerning the arithmetical algebra in the Greek and Indian civilizations.

5 Aydın Sayılı, *Mısırlılarda ve Mezopotamyalılarda Matematik, Astronomi ve Tıp* (Ankara: TTK Basımevi, 1966), 4–6, 45; George Sarton, *A History of Science: Ancient Science through the Golden Age of Greece* (New York: Dover Publications, 1952), 70–73; Bartel Leendert van der Waerden, *Bilimin Uyanışı* (Istanbul: Türk Matematik Derneği Yayınları, 1994), 93–122; Solomon Gandz, "The Origin and Development of the Quadratic Equations in Babylonian, Greek, and Early Arabic Algebra," *Osiris* 3 (1937): 551–55.

6 Jens Høyrup, "Old Babylonian 'Algebra' and What it Teaches us about Possible Kinds of Mathematics," *Ganita Bharati* 32, no. 1-2 (2010): 98, 109; George Sarton, *A History of Science*, 73; Waerden, *Bilimin Uyanışı*, 109.

In spite of Mesopotamian and Egyptian mathematic serving as a basis for Greek mathematics, and hence the prominence and priority of numerical methods, one has to concede that the geometrical methods prevail, especially in algebra, except for Diophantus' *Arithmetica* (third century CE).[7] Numerous reasons can be cited for this, but probably the most significant one is the basis of the analytical method in numbers and the non-correspondence of numbers to integers due to irrational and fractional numbers.[8] Unable to maintain the logical necessity they sought in non-integers, Greek mathematicians turned to geometrical methods and paved the way to what would later become geometrical algebra.

In his work, which represents the golden age of Indian algebra in the seventh century,[9] Brahmagupta, along with his explanation of the rules of calculation related to zero, presented three kinds of quadratic equations and another type of equation that he made by employing the notion of a negative number:

$$0 \times a = 0, \quad 0 \times 0 = 0, \quad \sqrt{0} = 0$$

$$\mathbf{1}.\ ax^2 + bx = c \qquad \mathbf{2}.\ bx + c = ax^2 \qquad \mathbf{3}.\ ax^2 + c = bx \quad and \quad px^2 + qx + r = 0$$

In addition to this, due to their recognition of negative and irrational numbers and notice of two roots for quadratic equations, Indian algebraists invented what is today called the "Indian method" by combining the solution of quadratic equations with the method of "summing up to square."[10] Consequently, Indian mathematicians made remarkable advances in algebra, particularly in analytical algebra, like the symbolization of zero, the development of the decimal positional number system, and convenient figures for nine numbers.[11]

7 Mainly taking "numerical analysis" as its subject without the use of variables in algebraic notation that we came to use today, he utilizes certain devices like the "substitution of the unknown with an added unknown," "algebraic shorthand," "the multiplication and division of exponents to the ninth degree," and "binary cubic calculations." While *Arithmetica* follows Mesopotamian algebra with respect to its analytical perspective, its lack of a generalized method and the denial of negative and irrational numbers as roots causes it to diverge in terms of including the types of indeterminate equations. For further details, cf. Diyūfanṭus al-Iskandarānī, *Ṣinā'at al-jabr*, translated by Qosṭā b. Lūqā, edited by Rushdī Rāshid (Cairo: al-Hay'at al-miṣriyyat al-'āmma li-l-kitāb, 1975), 7–20; Waerden, *Bilimin Uyanışı*, 461.

8 Bartel Leendert van der Waerden, *Geometry and Algebra in Ancient Civilizations* (Berlin and Heidelberg: Springer, 1983), 70–96; idem, *Bilimin Uyanışı*, 201.

9 The work *Brahmasphutasiddhanta* is not entirely concerned with mathematics; however, chapter 12 is on arithmetic and chapter 18 is on algebra. For the English translations of the parts dealing with arithmetic and algebra together with an assessment of the works of Brahmagupta and another noteworthy Indian mathematician Bhaskara II (1114–85), cf. Brahmagupta and Bhaskara, *Algebra, with Arithmetic and Mensuration*, trans. Henry Thomas Colebrooke (London: John Murray, 1817).

10 Carl B. Boyer, *Matematiğin Tarihi*, 251–3.

11 Colin Ronan, *Bilim tarihi* (Ankara: Tübitak, 2005), 212–13; Carl B. Boyer, *Matematiğin Tarihi*, 241–46.

2. The Foundation and Development of Arithmetical Algebra

The autonomy of the discipline of algebra is recounted in al-Khwārizmī's[12] *Kitāb al-mukhtaṣar fī al-jabr wa al-muqābala*. Although there was some input into algebra prior to al-Khwārizmī and other contemporaneous works similar to his, current research concurs that his work marks the beginning of this discipline,[13] for first he gives the arithmetical solutions of algebraic equations and then the geometrical proofs of the solutions. This pattern, which might be called the "Khwārizmian method," calls for the proper utilization of both methods. His successors, who stressed certain points and particular problems, devised various algebraic approaches/perspectives.

Thābit b. Qurra[14] (d. 288/901), the first of these successors, deserves to be mentioned here for even though he was a prominent figure in the perspective of geometrical algebra, he was the first one to supply a clearer differentiation of the methods of geometrical and arithmetical algebra. In his treatise *al-Qawl fī taṣḥīḥ masā'il al-jabr bi-l-barāhīn al-handasiyya*, he presented the geometrical conversion of quadratic equations via the equation,

$$\{x^2 + ax = b, \quad x^2 + b = ax \quad and \quad x^2 = ax + b\}$$

thus veritably fortifying the geometrical proofs, and tried to prove that thanks to the geometrical conversion for algebraic ways of solution, both methods obtained the same results. By noting the similarities in the ways that Euclidian geometrics and al-Khwārizmian algebra solved equations, he demonstrated the appli-

12 For further information on Khwārizmī's life, as well as the speculations concerning his name, works, and studies in Bayt al-ḥikma, cf. G. J. Toomer, "al-Khwarizmi," *DSB* VII, 358–65; Bartel Leenert van der Waerden, *A History of Algebra* (Berlin: Springer Verlag, 1985), 3–15; İhsan Fazlıoğlu, "Harizmî," *DİA* XVI, 224–27.

13 Some of the reasons can be spelled out as follows: (1) His statement of principles in a field that was hardly touched before him and his making the rules and methods of the new discipline useful not only for mathematicians, but also for clerks, merchants, judges, and officials. (That is why more than half of the book includes practical calculations.); (2) His grounding of an autonomous discipline by recognizing something more than a method for obtaining the unknown in algebra, hence preventing its being restricted to just a method; and (3) A conventional agreement in scholarly circles on taking the discipline's name from the title of his book. For further information on Khwārizmī as the founder of algebra as a field of study, cf. Rushdī Rāshid, *Riyāḍiyyāt al-Khwārizmī: Ta'sīs 'ilm al-jabr*, trans. Niqūlā Fāris (Beirut: Markaz dirāsāt al-waḥdat al-'arabiyya, 2010).

14 For further information on Thābit's calculation of infinitesimals for the first time in the history of mathematics in the Muslim world and his adaptation of the Pythagorean theorem so that it could be applied to all triangles, cf. Aydın Sayılı, "Sabit İbn Kurra'nın Pitagor Teoriminin Tamimi," *Belleten* 21, no. 88 (1957): 527–46. On the claims that Thābit was a supporter of Ḥajjāj b. Maṭar, who tried to establish a Greek school of mathematics at Bayt al-ḥikma, and that he supported similar attempts, cf. Waerden, *Bilimin Uyanışı*, 20.

cability of the geometrical way in terms of both the solution and the proof, which freed geometrical algebra of the need for calculation and set the stage for its complete isolation from arithmetical algebra.[15]

In his *Kitāb fī al-jabr wa-al-muqābala* (taking the title from al-Khwārizmī's work), Abū Kāmil Shujāʿ b. Aslam (235/850–317/930) established the rules for fractional calculations, after which he moved on to analyzing multi-variable linear equations and equations which includes operations with irrational numbers, and then on to several equations that he converted into quadratic equations. Thus, one can suppose that he expanded algebraic calculation to rational and irrational numbers and made advances in the theory of equations. In other words, he set the ground for the arithmetization of algebra.[16] The strongest proofs of his contribution to this project is his recognition of algebra as a discipline with the potential for growth in both scope and content, his continuous pursuit of different methods instead of being content with a single one for finding the variable, and his statement of algebra as an art full of inventions waiting to be made – one that required intelligence and attention. Clearly, he did not view it as a mimetic and mechanical science.[17]

Approximately 150 years after al-Khwārizmī, the mathematician al-Karajī of Baghdad (fourth/tenth–fifth/eleventh century) extended the attempts and studies of applying arithmetic to algebra in every respect. He even formulated it as a project that involved applying arithmetic subjects and certain algorithms to algebraic expressions and polynomials in particular (i.e., the arithmetization of algebra).

This undertaking by al-Karajī and his successors was regarded as one of the foremost projects of the contemporary history of algebra in the Muslim world. The goal of this perspective, which represented a departure from the predecessors' way, was the pursuit of new methods in order to obtain a more solid reconstruction of algebra with respect to its autonomy and features by refraining from using geometrical examples in algebraic operations.[18] Thus, al-Karajī took algebra out of Euclidian geometry's shelter and underscored the fact of its autonomy by rearranging its terms. His *al-Fakhrī* and *al-Badīʿ*, as part of this project, were subjected to

15 B. A. Rosenfeld and A. T. Grigorian, "Thabit Ibn Qurra," *DSB* XIII, 291; Rushdī Rāshid and Régis Morelon, *Mawsūʿa tārīkh al-ʿulūm al-ʿarabiyya* (Beirut: Markaz dirāsāt al-waḥdat al-ʿarabiyya, 1997), II, 468; Waerden, *Bilimin Uyanışı*, 18–19; İhsan Fazlıoğlu, "Sabit b. Kurre," *DİA* XXXV, 353–56.

16 Rāshid and Morelon, *Mawsūʿa tārīkh al-ʿulūm al-ʿarabiyya*, II, 469–70.

17 Martin Levey, "Abu Kamil," *DSB* I, 31.

18 For an extended assessment of al-Karajī's *ʿIlal ḥisāb al-jabr wa-al-muqābala* composed without employing any geometrical example, cf. Melek Dosay, *Kerecî'nin İlelü Hesab el-Cebr ve'l-Mukâbele Adlı Eseri* (Ankara: AKM, 1991), 29–56.

annotation, commentary, and studies by mathematicians until the seventeenth century. In fact, they occupied a central place in the field of algebraic calculation. The successors of al-Karajī, who can be dubbed the "renovator of algebra" – after all, he improved arithmetical algebra and thus paved the way for the new perspective – were al-Shahrazūrī, Ibn al-Turāb, Ibn al-Hishām, al-Samaw'al al-Maghribī, Ibn al-Khawwām, al-Tanūkhī, Kamāl al-Dīn al-Fārisī, Ibn al-Bannā' al-Marrākushī, Jamshīd al-Kāshī, al-Yazdī, 'Abd al-Majīd al-Sāmūlī, and Ibn Ḥamzat al-Maghribī.[19]

Al-Samaw'al al-Maghribī (d. 575/1180), the closest successor of al-Karajī, can be counted within the intellectual lineage that continued and seconded his predecessor's research program of arithmetizing algebra. However, the geometrical demonstrations and proofs contained within his studies show that he did not entirely discard the geometrical perspective, and thus contributed duly to both perspectives. Until the sixth/twelfth century, his *al-Bāhir fī al-jabr* was regarded as almost the summit that algebra could reach.[20]

Al-Maghribī first came up with the definition $\{x^0 = 1\}$ by conceptualizing the "algebraic exponent" and gave the rule for the formula $\{m, n \in Z$ such that $x^m . x^n = x^{m+n}\}$. His interest in the arithmetical operations of the roots and exponents of mononomials and polynomials led him to include zero in these operations and to explain the rules of multiplication for negative and positive numbers, thereby developing his understanding of negative numbers. In this context, he devised the definitions of $\{0 - a = -a \ and \ 0 - (-a) = a\}$, which enabled him to analyze the basic arithmetical operations for the division of mononomials and polynomials as well as to search for close approximations of fractions by polynomials. In addition, he dealt with calculating the square roots of polynomials with rational coefficients. The result of all of these efforts was the table for the division of polynomials and root subtraction, which could be considered a precursor to what today is called the "Ruffini-Horner method." This expansion of algebraic calculus in irrational numbers due to this method's enabling the easy solution of long and hard operations (e.g., the division of equations with twelve terms by four terms, or octonomials by trinomials) advanced the project of arithmetizing algebra.[21]

19 Rāshid and Morelon, *Mawsū'a tārīkh al-'ulūm al-'arabiyya*, 473; Rushdi Rāshid, *Tārīkh al-riyāḍiyyat al-'arabiyya bayn al-jabr wa-al-ḥisāb* (Beirut: Markaz dirāsāt al-waḥdat al-'arabiyya, 2004), 35; Aḥmad Salīm Sa'īdān, *Tārīkh 'ilm al-jabr fī al-'ālam al-'arabī* (Kuwait: al-Majlis al-waṭanī li-l-thaqāfa wa-al-funūn wa-al-ādāb, 1985), I, 83; Rushdī Rāshid, "al-Karaji," *DSB* VII, 242.

20 Sa'īdān, *Tārīkh 'ilm al-jabr fī al-'ālam al-'arabī*, 373; Adel Anbouba, "al-Samaw'al," *DSB* XII, 91; İhsan Fazlıoğlu, "Semev'el el-Mağribî," *DİA* XXXVI, 488-492.

21 Al-Samaw'al al-Maghribī, *al-Bāhir fī al-jabr*, edited by Ṣalāḥ Aḥmad and Rushdī Rāshid (Damascus: Wizārat al-ta'līm al-'ālī, 1972), 44–50.

The last active eastern mathematician who needs to be mentioned with respect to his contributions to arithmetical algebra is Sharaf al-Dīn Ṭūsī (sixth/twelfth century). Although regarded as a successor of ʿUmar Khayyām, one of the most prominent names in geometrical algebra, due the geometrical proofs and methods Ṭūsī offered in his *al-Muʿādalāt*,[22] his contribution to the development of analytical algebra is non-negligible. In this sense the theory of equations, which now occupies a more considerable place in algebra now thanks to his work, stood at its heart. Ṭūsī brought together the geometrical study of equations and the numerical solutions in the content of this theory and solved the problem of providing a separate solution for each equation. Not only did the curves he utilized lead to the invention of position analysis, but he also particularly pioneered the methodological inquiry of "cubic polynomials" by "derivative equations." Not content with applying algorithms in the field of numerical solutions, he proposed the expression "derivative of polynomial" and tried to verify these algorithms with the notion of "dominant polynomials."[23] However, his greatest contribution to arithmetical algebra was the method of tabulation seen at al-Samaw'al al-Maghribī's work that enabled one to find the root of polynomial equations more efficiently, namely, the easier and faster way to the solution set of higher-order equations.

As regards the mathematicians of the western Muslim world, three names stand out in the context of this article and algebraic studies. The first one, in chronological order, is Ibn Yāsamīn (601/1204–05), who wrote the discipline's first versified work. In it, he summarizes the rules of equation solutions as proposed by al-Khwārizmī in approximately 40 couplets in his famous *al-Urjūzat al-yāsamīniyya fī ʿilm al-jabr wa-al-muqābala*, which can be considered a cornerstone for its ability to show the effective communication of analytical algebra through verse. The work, as an example of the western mathematical school and one subjected to numerous commentaries and glosses over the centuries,[24] is among the first works in verse of pure arithmetical algebra. The commentaries on Ibn Yāsamīn by mathematicians who played important roles in the Ottoman mathematical lineage, like Ibn

22 For the edited text of the treatise, cf. Rushdī Rāshid, *al-Jabr wa-al-handasa fī al-qarn al-thānī ʿashar: muʿallafāt Sharaf al-Dīn al-Ṭūsī* (Beirut: Markaz dirāsāt al-waḥdat al-ʿarabiyya, 1998), 433-551.

23 Rāshid and Morelon, *Mawsūʿa tārīkh al-ʿulūm al-ʿarabiyya*, 488.

24 For further information on this work and the edited texts, exegeses, commentaries, and glosses on the author's other works, cf. Ibn Yāsamīn, *Manẓūmāt Ibn Yāsamīn fī aʿmāl al-jabr wa-al-ḥisāb*, edited by Jalāl Shawqī (Kuwait: Mu'assasat al-Kuwayt li-l-taqaddum al-ʿilmī, 1988). For the excerpts and lists of all manuscript copies, commentaries, and glosses on *al-Urjūzat al-yāsamīniyya fī ʿilm al-jabr wa-al-muqābala*, cf. Jalāl Shawqī, *al-ʿUlūm al-ʿaqliyya fī al-manẓūmāt al-ʿarabiyya* (Kuwait: Mu'assasat al-Kuwayt li-l-taqaddum al-ʿilmī, 1990), 220–61.

al-Hā'im and Sibṭ al-Mardīnī,[25] and the circulation of these commentaries among scholarly circles, are remarkable for indicating both the eastern and western roots and influences of arithmetical algebra on mathematics within the Ottoman realm.

Ibn al-Bannā' al-Marrākushī (721/1321–654/1254), the second mathematician and the founder of the "Bannā' school," was influential in spreading pure arithmetical algebra in the Maghreb via his *Kitāb al-jabr wa-al-muqābala*,[26] which was entirely devoted to algebra, and chapter 2, section 2 ("al-jabr wa-al-muqābala")[27]of *Talkhīṣ aʿmāl al-ḥisāb*, his renowned book of practical calculation. Although there is an ongoing debate as to whether the former work is a sort of compendium along the lines of Abū Kāmil's book of algebra, and especially its commentary by Abū al-Qāsim al-Qurashī, the analysis of calculation in separate sections for the known and the unknown in Ibn al-Bannā''s work, the solution of quartic equations via the methods of identity and factorization, and the omission of geometrical demonstration and proof in any fashion indicate that the author presented the most comprehensive algebraic knowledge of the Maghrebi line of mathematics.[28]

The third and final mathematician is al-Qalaṣādī (d. 891/1486), a member of "Bannā' school" of the later period. Although he focused on arithmetic rather than algebra, he is mentioned here due to the solid presence of arithmetical and algebraic notation, earlier forms of which first appeared 100 years prior to him with Ibn al-Bannā', in his works.[29] In addition to the ability to operate with higher-order equations, the common instrument of defense for mathematicians of the arithmetical algebra perspective against the supporters of geometrical algebra is that symbolic representation, which eased the teaching, learning, and transmission of algebra over the centuries, is effectively administered only by arithmetical algebra.

25 Both commentaries were published in critical editions: Ibn al-Hā'im, *Sharḥ al-Urjūzat al-yāsamīniyya fī al-jabr wa-al-muqābala*, edited by ʿAbd al-Jawād al-Mahdī (Tunis: Manshūrāt al-jamʿiyyat al-Tūnisiyya li-l-ʿulūm al-riyāḍiyya, 2005); Sibṭ al-Mardīnī, *al-Lamʿat al-Mardīniyya fī sharḥ al-yāsamīniyya*, edited by Muḥammad Suwaysī (Kuwait: al-Majlis al-waṭanī li-l-thaqāfa wa-al-funūn wa-al-ādāb, 1983).

26 For the text and exegesis, cf. Aḥmad Salīm Saʿīdān, "Kitāb al-jabr wa-al-muqābala li-Ibn al-Bannā' al-Marrākushī," in *Tārīkh ʿilm al-jabr fī al-ʿālam al-ʿarabī*, II, 505–85.

27 Ibn al-Bannā' al-Marrākushī, *Talkhīṣ aʿmāl al-ḥisāb*, edited by Muḥammad Suwaysī (Tunis: al-Jāmiʿat al-Tūnisiyya, 1969), 73–77.

28 İhsan Fazlıoğlu, "İbnü'l-Bennâ el-Merrâküşî," *DİA* XX, 532. The author of the article gives the title as *Kitāb al-uṣūl wa-al-muqaddamāt fī al-jabr wa-al-muqābala*.

29 Another prominent work on notation is the commentary on Ibn al-Bannā''s *Talkhīṣ* by Ibn al-Qunfudh (d. 772/1370), entitled *Ḥaṭṭ al-niqāb ʿan wajh al-ʿamal bi-al-ḥisāb*. However, it remains disputed whether the author improved on the symbols employed in the manuscripts that he utilized while composing his own work or invented his own. On the question of the timing of algebraic notation, whether initiated by the eastern mathematicians or by the western mathematicians at a later period, cf. Salih Zeki, "Notation algébrique chez les Orientaux," *Journal Asiatique* seri 9, no. 11 (1898): 35–52. For its translation into Turkish, cf. Remzi Demir, "Salih Zeki Bey'in Journal Asiatique'de Yayımlanan 'Notation algébrique chez les Orientaux' Adlı Makalesi," *OTAM*, no. 15 (1977): 333–53.

After these eastern and western mathematicians, the mathematicians of Anatolia who lived in between and their contributions to arithmetical algebra have to be mentioned. But due to the limits of this article, only Ibn al-Khawwām (d. 724/1324) and his *al-Fawā'id al-bahā'iyya fī al-qawā'id al-ḥisābiyya*,[30] and Niẓām al-Dīn Nīsābūrī (d. post 727-30/1326-30) and his *al-Shamsiyya fī al-ḥisāb* are noted in this regard.[31] Even though neither of them were Anatolians, they nevertheless achieved remarkable influence via their works and the institutions of education. The probable reason for their influence is as follows: The renowned scholar Naṣīr al-Dīn Ṭūsī (d. 673/1273) built an observatory at Maragha, south of Tabriz, in the seventh/thirteenth century with the patronage of the Ilkhanid ruler Hulegu and turned it into a school of astronomy and mathematics where research, publishing, and teaching were undertaken together with prominent mathematicians and astronomers. Many later scholars conveyed various studies of Ibn al-Khawwām, a full member of the school, and Niẓām al-Dīn Nīsābūrī, an associate member principally in mathematics, to Anatolia together with the school's entire accumulated knowledge. Ibn al-Khawwām's contribution is his addition of the multiplication and division of polynomials, the addition of arithmetic sequences to his book's chapter on algebra, and his organization of the chapter into two sections: theoretical input and applied problems. Nīsābūrī analyzed the examples of finding the root of polynomials and the fourth-order irrational roots of rational numbers in the section on algebra and sought to improve arithmetical algebra's performance in operations involving higher-order equations.[32]

Arithmetical algebra reached its peak during the fourteenth and fifteenth centuries. However, before discussing that topic we should assess the place and importance of *al-Muqni'* and why Ibn al-Hā'im composed this comprehensive work that is deservedly called the "encyclopedia of algebra" and the "dictionary of algebra."

30 For detailed information on the author and his book of mathematics, cf. İhsan Fazlıoğlu, "İbn el-Havvâm ve Eseri Fevâidü'l-Bahâiyye fi'l-Kavâidi'l-Hisâbiyye: Tenkitli Metin ve Tarihi Değerlendirme," (Unpublished MA thesis, Istanbul University, 1993).

31 For detailed information on the author and his book of mathematics, cf. Elif Baga, "Nizâmuddin Nisâbûrî ve eş-Şemsiyye fi'l-Hisâb Adlı Matematik Risâlesinin Tahkik Tercüme ve Tarihi Bir Değerlendirmesi," (Unpublished MA thesis, Sakarya University, 2007).

32 For further information on the history of mathematics, particularly algebra, in Ottoman lands, cf. Elif Baga, "Osmanlı Klasik Dönemde Cebir," (Unpublished PhD diss., Marmara University, 2012).

II. The Genre of Versified Works in The Muslim World

Although the words for verse (*naẓm*, *manẓūma*; from the root *n-ẓ-m* [to lay pearls, to order]) are generally used for poems and poetry, verse is regarded as a didactical genre of poetry that possesses no sensibility and imagery, but does possess the elements of meter and rhyme.[33] It diverges from poetry in that it can be expressed in prose, is driven by an objective plot, and the prominence of the real meaning; however, it is as old as poetry. As this genre existed in ancient India and Greece long before the Arabs, it is held to have had an educational purpose: making it easier for the pupil to memorize the lesson to be learned. Early Arabic examples with didactic quality, variously called *qaṣīda* or *rajaz*, were first attested to during the pre-Islamic age. However, their reception as two distinct genres for differing poetical techniques, as well as their maturation, coincided with the Umayyad and Abbasid periods. Khalīl b. Aḥmad's[34] (175/791) incorporation of *rajaz* into the prosodic system and the formation of a particular meter set the stage for longer emphatically didactical pieces called *urjūza*.[35]

Scientific activities and the composition of works quickened under the Abbasids and, later on, resulted in the emergence and development of pedagogical techniques designed for easy and fast learning in educational institutions. Different from prose only for its rhyme and ease of memorization, it was applied to all bodies of learning from linguistics to catechism, medicine to mathematics, and contributed significantly to both the formation of disciplines and the efficiency of education. Rather than *qaṣīda*, *urjūza* was the more popular form the birth of Islamic civilization until the thirteenth/nineteenth century. The probable reason for this could be that the second line[36] of all couplets followed a single rhyme in *qasīda*, whereas it sufficed to rhyme lines within a couplet in *urjūza*. In short, the fact that the rhyme was stricter in *qaṣīda* could have led the authors to prefer the latter. However, Ibn al-Hā'im composed *al-Muqni'* in *qaṣīda* form, as if to prove to his peers that he was up to this tough task. Maybe for his use of a single rhyme throughout the text, it was considered among those works that could be easily and

33 İsmail Durmuş, "Şiir," *DİA*, XXXIX, 144.

34 Combinatorics, which can be defined as a technique of counting, first emerged in the Muslim world in the linguist Khalīl b. Aḥmad's studies of deriving meaningful words from Arabic letters. For an extended explanation of the combinatorial analysis with applications in fields like algebra first and then language and philosophy, cf. Rüşdi Râşid, "Matematik," *DİA* XXVIII, 132; Ahmed Djebbar, "Combinatorics in Islamic Mathematics," *Encyclopaedia of the History of Science, Technology and Medicine in Non-Western Cultures*, ed. Helaine Selin (Dordrecht: Kluwer, 1997), 230–32.

35 Kemal Tuzcu, "Klasik Arap Şiirinde Didaktik Şiirler," *Ankara Üniversitesi DTCF Dergisi* 47, no. 2 (2007): 148–50.

36 The name of each part that made up the couplet.

quickly memorized, and therefore popularly memorized by students in the educational institutions.[37]

Concerning the versified works of mathematics composed in the disciplines of numerology, arithmetic, algebra, measurement, and geometry, one can first suggest that extant works date from sixth/twelfth–thirteenth/nineteenth centuries. Of the approximately ninety works in this genre, most of them are on arithmetic and algebra. Sorting them according to their currency (i.e., the number of extant copies and the studies on them) results in the following chart:

i. Ibn Yāsamīn's (601/1204) versified *al-Urjūzat al-yāsamīniyya fī al-jabr wa-al-muqābala* circulated the most: almost 50 copies of the text, around 180 copies in commentaries by 22 different commentators,[38] and 33 glosses.

ii. Ibn al-Hā'im's (815/1412) widely circulated *al-Muqni' fī al-jabr wa-al-muqābala*: approximately 25 copies of the text, 90 copies in commentaries by 7 commentators, and 4 glosses.

iii. Ibn Ghāzī al-Miknāsī's (919/1513) *Munyat al-ḥussāb fī 'ilm al-ḥisāb*, the versified form of Ibn al-Bannā' al-Marrākushī's (721/1321) *Talkhīṣ a'māl al-ḥisāb* with additions: roughly 15 copies of the text, and around 30 copies in commentaries by 4 different commentators.[39]

Finally, a statistical distribution of the extant versified works composed in the field of mathematical sciences can illustrate the balance across the disciplines.[40] Accordingly:

1. Arithmetic (including fractional calculations): 40%
2. Algebra (including root calculations): 20%
3. Numerology: 10%
4. Science of measurement: 9%
5. Geometry: 6%
6. Counting by fingers: 6%
7. Mathematics (General mathematical works in verse): 5%
8. Science of shares in inheritance: 4%.

37 Sonja Brentjes, "Teaching the Mathematical Sciences in Islamic Societies: Eighth Seventeenth Centuries," in *Handbook on the History of Mathematics Education*, edited by Alexander Karp and Gert Schubring (New York: Springer, 2014), 96.

38 The most popular commentaries of *al-Urjūzat al-yāsamīniyya* were Sibṭ al-Mardīnī's *al-Lam'at al-Mardīniyya fī sharḥ al-yāsamīniyya* and Ibn al-Hā'im's *Sharḥ al-Urjūzat al-yāsamīniyya fī al-jabr wa-al-muqābala*.

39 One of the commentators is the author himself. His commentary, *Bughyat al-ṭullāb fī sharḥ Munyat al-ḥussāb,* was more renowned than the versified work itself and was published twice (i.e., Morocco [1899] and Aleppo [1983]). For further information on the work's content and the copies of the commentary, cf. Jalāl Shawqī, *al-'Ulūm al-'aqliyya fī al-manẓūmāt al-'arabiyya*, 288–96.

40 I computed the figures based on the data provided by Jalāl Shawqī's *al-'Ulūm al-'aqliyya fī al-manẓūmāt al-'arabiyya*. (pp. 211–338).

III. The Author: Ibn al-Hā'im

The author, Shihāb al-Dīn Abū al-ʻAbbās Ibn al-Hā'im, was born in Qarafa, Egypt, in 753/1352-53 or 756/1355. He studied jurisprudence, Arabic linguistics, the rules of inheritance and arithmetic, and eventually became a prominent expert in all of them. He spent the first half of his life taking courses from tutors in Cairo and in self-study, and the second half of his life in Jerusalem teaching students and composing works. He finally passed away there in 815/1412, having spent the last twenty years of his life as a college professor and director being close the students led him to produce works with nuanced pedagogical intent. Especially the classification, system, language, and exposition he employed in his mathematical studies, including the study of inheritance, can be taken as evidence for this assumption. Considering the subjects he dealt with in his extant works, which number approximately forty, one can say that he was first a mathematician and then a jurisprudent.

The references to Nūr al-Dīn al-Jilāwī in some of his mathematical studies[41] hint that this figure was one of his mentors. We may also suppose that he took courses from Chief Mufti Sirāj al-Dīn ʻUmar b. Balqīnī (d. 805/1403), Sheikh Jalāl al-Dīn Amyūṭī (d. 790/1388), Ibn al-Khātim, and ʻAbd al-Raḥmān b. Ḥusayn al-ʻIrāqī.[42] Given his long years of teaching, one may suppose that he had numerous students, the most notable of whom was Ibn Ḥajar al-ʻAsqalānī.[43]

A prolific author, Ibn al-Hā'im composed roughly eighteen mathematical works[44] on algebra, arithmetic, and inheritance. Most of these works circulated for almost two centuries, especially in the circles of scholars and colleges of the Ottoman realm, and further commentaries and glosses were drafted.[45] Some of the notable works are the arithmetic encyclopedia *al-Maʻūna fī ʻilm al-ḥisāb al-hawā'ī*; its compendium *al-Wasīla ilā ṣināʻat al-hawā'ī*, *al-Lumaʻ fī al-ḥisāb*, *Murshidat al-ṭālib ilā asnā al-maṭālib*; and its compendium *Nuzhat al-nuẓẓār fī ṣināʻat al-ghubār*, *al-Ḥāwī*

41 Ibn al-Hā'im, *el-Mumtiʻ fī sharḥ al-Muqniʻ*, MS Chester Beatty 3881, 2a (autograph copy); idem, *al-Muqniʻ fī al-jabr wa-al-muqābala*, Süleymaniye Library, MS Reisülküttab 1191, 59b; idem, *al-Maʻūna fī ʻilm al-ḥisāb al-hawā'ī*, edited by Khuḍayr ʻAbbās Muḥammad al-Munshidāwī (Baghdad: Dār al-āthār wa-al-turāth, 1988), 28; idem, *Shubbāk al-munāsakhāt*, King Saud University MS 1044, 1b.

42 Ibn al-Hā'im, *al-Maʻūna fī ʻilm al-ḥisāb al-hawā'ī*, 28; idem, *al-Tibyān fī tafsīr gharīb al-Qur'ān*, edited by Fatḥī Anwar Dābūlī (Cairo: Dār al-ṣaḥāba, 1992), 23; idem, *al-Fuṣūl fī al-farā'iḍ*, edited by ʻAbd al-Muḥsin b. Muḥammad b. ʻAbd al-Muḥsin Munīf (Riyadh: n.p., 1994), 11–13.

43 For further information on his students, cf. Ibn al-Hā'im, *al-Maʻūna fī ʻilm al-ḥisāb al-hawā'ī*, 32–34.

44 It is not appropriate to give an exact number, for some of his works have survived only in part, whereas others did not survive at all. For further information, cf. İhsan Fazlıoğlu, "İbnü'l-Hâim", *DİA*, XXI, 63-65.

45 Sonja Brentjes, "Teaching the Mathematical Sciences in Islamic Societies," 106; Cevat İzgi, "Osmanlı Medreselerinde Aritmetik ve Cebir Eğitimi ve Okutulan Kitaplar," *Osmanlı Bilimi Araştırmaları* 1 (1995): 145.

fī al-ḥisāb which is a compendium of Ibn al-Bannā'’s *Talkhīṣ aʿmāl al-ḥisāb*, *Sharḥ al-Urjūzat al-yāsamīniyya fī al-jabr wa-al-muqābala*, and *al-Fuṣūl fī al-Farā'iḍ*.

The probable reasons for his works’ popularity are (1) his preference for entirely analytical solutions in both Indian and mental computation, and in algebra in line with the trends of the day; (2) his writing style that accorded to his students’ needs, which he knew well due to his long years of teaching in colleges; and (3) his reproduction of his works in lengthy-medium-concise forms to suit the interests of a variety of readers. A fine example of the last is the trilogy of his work *al-Muqniʿ*, its commentary *al-Mumtiʿ*, and the compendium *al-Musriʿ*.

IV. Original Works: al-Muqniʿ and its Commentary al-Mumtiʿ

Ibn al-Hā'im’s *al-Muqniʿ fī al-jabr wa-al-muqābala*[46] covers the disciplines of algebra and equation balancing in *qaṣīda* form. It consists of 59 couplets. *Al-Mumtiʿ fī sharḥ al-Muqniʿ* is his own commentary on this *qaṣīda* of algebra. Yet again, the author composed the compendium a*l-Musriʿ fī sharḥ al-Muqniʿ*[47] revising his commentary due to reader demand. Some sources and library catalogues cite *al-Musmiʿ* as a compendium of *al-Musriʿ*. I was unable to verify whether it was a compendium of *al-Musriʿ* or mistaken for *al-Musriʿ* itself due to the unavailability of copies.[48] Given its comprehensive coverage of the contemporaneous algebraic knowledge, the extensive *al-Mumtiʿ fī sharḥ al-Muqniʿ* was subjected to further analysis.

1. Al-Muqniʿ Fī al-Jabr Wa-al-Muqābala

The most popular and widely circulated mathematical work after Ibn Yāsamīn’s *al-Urjūzat al-yāsamīniyya fī al-jabr wa-al-muqābala* is Ibn al-Hā'im’s *al-Muqniʿ fī al-jabr wa-al-muqābala*. Not treated in a separate work, there are approximately 25 extant copies.[49] Starting to introduce the work briefly by its content, after a six-cou-

46 The work appears to still be in manuscript form, except for the excerpt in Jalāl Shawqī’s book. For further information on *al-Muqniʿ*, cf. idem, *al-ʿUlūm al-ʿaqliyya fī al-manẓūmāt al-ʿarabiyya*, 268–82.

47 The autography copy of *al-Musriʿ fī sharḥ al-Muqniʿ*, completed in al-Aqsa Mosque five days after the commentary *al-Mumtiʿ* in 810/1408, is Ahmadiya Library MS 107 in Mosul. The copies of the work are MS Laleli 3747 and 3752. For further information on al-Musriʿ, cf. Shawqī, *al-ʿUlūm al-ʿaqliyya fī al-manẓūmāt al-ʿarabiyya*, 272–73.

48 Ibid., 273–74.

49 For *al-Muqniʿ*’s initial and final couplets, and for further information on the text’s manuscript copies, commentaries and glosses, and their manuscript copies, cf. Shawqī, *al-ʿUlūm al-ʿaqliyya fī al-manẓūmāt al-ʿarabiyya*, 268–83; *al-Maʿūna fī ʿilm al-ḥisāb al-hawā'ī*, 43–44; Boris A. Rosenfeld and Ekmeleddin İhsanoğlu, *Mathematicians, Astronomers, and Other Scholars of Islamic Civilizations*, 246.

plet introduction, it comprises five sections: the "name, degree, and exponents of the unknown kinds," "addition and subtraction," "multiplication and division," "six algebraic equations," and "chapter."[50]

After the introductory prayer couplet, he refers to his mentor al-Jilāwī and the dignity of algebra and states that only the elect with an aptitude for mathematical sciences would turn to this discipline. Moreover, he suggests that the *qaṣīda* contains the essence of algebra and would suffice for the prudent.[51] Under the first heading, he mentions the basic terms of algebra like root/x (*al-jadhr*), square/x^2 (*al-māl*), and cube/x^3 (*al-ka'b*) as well as the derivative terms. In addition, he refers to an important discussion by pointing out the finer distinctions between the variable (*al-shay'*) and the root, and the cube and the three times the variable (*al-muka'ab*).[52] The second section, consisting of five couplets, considers the nature of addition and subtraction with algebraic terms. The third section discusses the algebraic application of multiplication and division in basic calculus. In the fourth section, which lies at the heart of algebra, the author mentions the six types of equations obtained by the binary and trinary combinations of number (*al-'adad*), variable (*al-shay'*), square (*al-māl*), and the separate solutions of these types of equations. The final section, which is more like a continuation of the previous one, concerns those problems/equations that do not fit into the aforementioned six types and how to convert them.

Since further characteristics of *al-Muqni' fī al-jabr wa-al-muqābala*'s content will be explicated with respect to its commentary *al-Mumti'*, only the text's formal characteristics will be dealt with here. Thus, the work consists of fifty-nine couplets written in the "long meter" (*baḥr al-ṭawīl*). It rhymes with the letter "*lām*," and thus the work is also called *Lāmiyya Ibn al-Hā'im*. Moreover, the author conveys the place and date for its finalization in the last two couplets. But he picked a different course for it:

وأبياتها تسع وخمسون انشئت   بالأقصى وشهر اليمن فهي تطاول

ربيع من العام الذي ضبط عدّه   بدال وضاد فالبناء متكامل[٥٣]

The couplets of the *qaṣīda* are fifty-nine, composed in al-Aqsa in the month of fortune, it prolongs.

A quarter of the year number of which marked by "*dāl*" and "*ḍād*" and then the *qaṣīda* is finished.

50 Ibn al-Hā'im, *al-Muqni' fī al-jabr wa-al-muqābala*, Süleymaniye Library, MS Reisülküttab 1191, 59a-61b.

51 Ibid., 59b.

52 Ibid., 59b-60a.

53 İbnü'l-Hâim, *el-Mukni'*, vr. 61b.

According to these couplets, Ibn al-Hā'im finished *al-Muqnī* in al-Aqsa Mosque during the month of Rabī' I, which falls during the first quarter of the year (*rabī' min al-'ām*) and is called the "month of fortune" (it is the Prophet's birth month), in the year 804/1401, as "*dāl*" stood for four and "*ḍād*" for 800 in numerology.

2. al-Mumti' fī Sharḥ al-Muqni'

The most comprehensive of the seven commentaries on *al-Muqni'*, two of which are anonymous, is Ibn al-Hā'im's *al-Mumti'*. Not treated in a separate study until now,[54] there are around twenty extant copies (including the autograph [a manuscript handwritten by its author]).[55] The author finished it six years after *al-Muqni'* in al-Aqsa Mosque on 13 Jumāda I 810 / January 12, 1407.[56]

a. Content

Along with recounting the objectives of the study of algebra in the introduction of *al-Mumti'*, Ibn al-Hā'im also provides content matter. Hence, the objectives and the author's writing style to achieve its goals are as such:

i. The explanation of variable, square, cube, square of square (*māl al-māl*), square of cube (*māl al-ka'b*), cube of cube (*ka'b al-ka'b*), and further terms in use by the experts, and their degrees and exponents.

ii. The explanation of operations in addition, subtraction, multiplication, and division and taking the root with algebraic expressions.

iii. The explanation of the six types of equations into which any equation can be converted and the various methods of obtaining their solution sets.

iv. The nature of manipulating the equation until it turns into one of the six types of equations and the methods to be employed in order to bring this conversion about.

In his opinion, the first article can be regarded as an introduction and the last article as the conclusion of the second and third articles. Thus, *al-Mumti'* consists of

54 I will soon publish the edited text and the translation based on the autograph.

55 Throughout the article, the autograph copy at MS Chester Beatty 3881 will be used. For other copies, cf. Shawqī, *al-'Ulūm al-'aqliyya fī al-manẓūmāt al-'arabiyya*, 271; Ibn al-Hā'im, *al-Ma'ūna fī 'ilm al-ḥisāb al-hawā'ī*, 28; Rosenfeld and İhsanoğlu, *Mathematicians, Astronomers, and Other Scholars of Islamic Civilizations*, 246.

56 Ibn al-Hā'im, *al-Mumti' fī sharḥ al-Muqni'*, MS Chester Beatty 3881, 68b.

an introduction, two chapters, and a conclusion.[57] The content matter provided by the author is too brief and the commentary contains no subheadings; however, the text is arranged in articles like "advices," "commands," "issues," and "forms." I will suggest below a more detailed content matter under the appropriate titles.

Introduction: *Invocatio dei*, praise to god and the prophet, the reason of composition, the title, the definition of algebra, reference to the discipline's founder Muḥammad b. Mūsā al-Khwārizmī (third/ninth century). (1b-2a)

The Names of the Unknown Types, Their Degrees and Exponents (2b-9b)

Addition and Subtraction with Algebraic Expressions

Problem 1: Addition and subtraction on the one side (9b-10a)

Problem 2: Addition and subtraction on the opposite side (10a-12a)

Problem 3: Subtraction with a negative term (12a-13a)

Problem 4: On declaring the method for cancelling out the negative term on one or both sides of the equation (13a-14b)

Multiplication with Algebraic Expressions

Part 1: Multiplication of a number with a kind (15a-15b)

Part 2: Multiplication of a kind with a kind (15b-18b)

- Multiplication of a mononomial with a mononomial
- Multiplication of a mononomial with a polynomial
- Multiplication of a polynomial with a polynomial

Part 3: Multiplication with a negative term (18b-21b)

Part 4: Multiplication with division (21b-23b)

- Multiplication of a fraction with a non-fraction
- Multiplication of a fraction with a fraction

Part 5: Multiplication with division and a negative term (23b-24a)

57 Ibid., 2b.

Division with Algebraic Expressions

Arrangement according to the number of terms (24a-26b)

- Division of a monomial by a mononomial
    - Division of a kind by a kind
    - Division of a number by a kind
    - Division of a kind by a number
- Division of a polynomial by a mononomial
- Division of a monomial by a polynomial
- Division of a polynomial by a polynomial

Arrangement according to the presence of negative term and/or fraction in the dividend and divisor (26b-29a)

- Division of a negative term by a positive term
- Division of a fractional term by a non-fractional positive term
- Division of a fractional and negative term by a non-fractional positive term
- Division of a non-fractional positive term by a fractional term
- Division of a non-fractional positive term by a fractional and negative term
- Division of a negative term by a fractional term
- Division of a negative term by a fractional and negative term
- Division of a fractional term by a fractional term
- Division of a fractional and negative term by a fractional term
- Division by a negative term or compounds of number and kind or two kinds and more
- Division of a non-fractional positive term by a fractional and divisional term
- Division of a fractional and negative term by a fractional and negative term

Taking the Square Root of a Mononomial and a Polynomial

- Taking the square root of a mononomial (29a-29b)
- Taking the square root of a polynomial (29b-30b)
    - There are no square roots to polynomials with an even number of terms
    - The square root of polynomials with an odd number of terms
- Induction (31a-32a)

Six Algebraic Equations

- Simple equations (32a-37b)
- Mixed equations (37b-42a)
- Warnings
  - The method to form a mixed equation by assuming two perfect squares (42a-42b)
  - The cause of the first type of mixed equation (42b-43a)
  - The cause of the second type of mixed equation (43a-44a)
  - The cause of the third type of mixed equation (44a-44b)
  - The formula to find the perfect square in the first mixed equation (44b-45a)
  - The formula to find the perfect square in the third mixed equation (45a-45b)
  - The formula to find the perfect square in the second mixed equation (45b-46b)
  - The formula to convert the first mixed equation to first or third simple equation (46b-47a)
  - The formula to convert the second mixed equation to first or third simple equation (47a-47b)
  - The formula to convert the third mixed equation to first or third simple equation (47b-48b)

The Nature of Converting Equations to One of Six Types

- The conversion of the equation by completion and deletion (48b-51b)
  - The First method/formula
  - The Second method/formula
- The formulas to solve mixed equations without completion and deletion (51b-53a)
- The formulas to find square in mixed equations without completion and dee letion (53a-54b)

Appendix: on the Unlimitedness of Algebraic Equations (54b-57a)

- Examples for the unlimitedness of simple equations (57a-57b)
- Examples for the unlimitedness of mixed equations (57b-58b)
- The solution method if the exponents of the variables in the equation are not consecutive (58b-59b)
- The solution method if the exponents of the variables in the equation are consecutive (59b-60a)
- Quotations from the commentary of Yāsāmīnī
  - Article 1: Conditions for the possibility of the equation (60b-61b)
  - Article 2: The givens of an equation (61b-62b)
  - Article 3: The nature of manipulating the equation (62b-66a)
- Miscellaneous problems (66a-68b)

b. Main Features of the Work

i. Extended commentary: First of all, it needs to be stated that the work is composed as an extended commentary. Thus, Ibn al-Hā'im goes to nooks and crannies of the text with the ease of commenting on his own work, even going into such linguistic details as specifying the words' spelling and explaining what they mean.

لفخر الزمان المنتمى لجلاوة   على عليه سحب جود هواطل

و«السحب» جمع سحابه، و«الجود» بفتح الجيم المطر الفزير، و «الهواطل» نعت للسحب وهو جمع هاطلة من الهطل وهو تتابع المطر والدمع وسيلانه.

"Suḥub" is the plural of *saḥāba* and "jawd" – with the inflection of *jīm* – is downpour (*al-fazīr*). "al-Hawāṭil" is the adjective of *suḥub* and the plural of *hāṭila* from the root *al-haṭl*. It is the fall and flow of rain and teardrops (*al-dama'*).

ii. Robust conceptual analysis: What initially catches one's attention in this work is the explanation of all concepts/terms used throughout algebra's history by nuances and examples. In this sense *al-Mumti'* is also a "dictionary of algebra," for the greater part listed under the first heading, "the names of the unknown types, their degrees and exponents," is presented according to a certain method of classification. The unknown types are first classified as primary and secondary vis-à-vis their names, exponents, and degrees, and then further subdivided within these titles.[58] To exemplify the precision of nuances in algebraic terms:

58 Ibid., 2b-9b.

إن المال يرادفه المربع، والمجذور عند من أطلق المال على المعلوم والمجهول. والمسطح والسطح والبسيط أعم من كل منها لأن المسطح ما قام من ضرب عدد في عدد سواء كان متساويين ام متفاضلين معلومين ام مجهولين ام مختلفين، وكذلك السطح والبسيط، وكذلك الضلع أعم من الجذر. إذ كل جذر ضلع، وليس كل ضلع جذرا كما أن كل مال ومربع ومجذور، مسطح وسطح وبسيط من غير عكس كلي.[59]

For those who apply "māl" to the known and the unknown, "murabba'" and "majdhūr" are identical, "musaṭṭaḥ," "saṭḥ," and "basīṭ" are more general than all. Since "musaṭṭaḥ" is what comes out of the multiplication of number by number, it is true irrespective of if (those numbers are) equal, consecutive, known, or unknown/variable. The generality of "dil'" more so than the "jadhr" is similar. While it holds that all "māl," "murabba'" and majdhūr" are "musaṭṭaḥ," "saṭḥ," and "basīṭ," but not vice versa, all "jadhr" are "dil'" but not all "dil'" are "jadhr."

Another matter concerning algebraic concepts is the ambiguous use of "shay'" and "jadhr" ever since the discipline's inception. Although it might be understandable that the concepts and their contents were neither clearly nor precisely demarcated in the earlier centuries, the emergence of some consensus later on might be expected. Responding to this expectation, Ibn al-Hā'im clarifies the issue:

فنبه بالبيت على أن بينهما عموما وخصوصا من وجه وهو المختار، لان كل أمرين اجتمعا في محل صدقا وانفرد كل منهما عن الآخر بالصدق في محل آخر كانا كذلك، فاذا فرض المجهول شيأ وضرب في مثله: فالمضروب شيء وجذر فهذا محل تصادقهما. واذا لم يضرب في مثله، فلا يسمى جذرا فهذا محل انفراد الشيء بالصدق عن الجذر. واذا ضرب عدد معلوم في مثله، فالمضروب جذر ولا يسمى شيأ في الاصطلاح المشهور فهذا محل انفراد الجذر بالصدق عن الشيء، ويفاصل بالمهملة اي افتراق.[60]

He noted a partial interchangeability between the two (the variable and the root) in the couplet, and it is the preferred usage. For it applies to two that can occupy the same place and are differentiated by occupying different places. **Assuming that the unknown is the variable and multiplied by the same, what is multiplied is the variable and the root and it is where they occupy the same. If it (the unknown) is not multiplied by the same, it would not be named root. It is where the variable differentiates from the root.** If a known number is multiplied by the same number, what is multiplied is the root, and would not be called the variable in common terminology, **it is where the root differentiates from the variable.**

59 Ibid., 4a.
60 Ibid., 4b-5a.

In addition to this significant distinction, the author sets forth the similarities and differences of “kaʿb” and mukaʿab,” “manāzil” and “marātib,” and “ḍurūb-i sitte” and “mesā’il-i sitte.”[61]

iii. The pedagogical principle of moving from the simple to the complex: A characteristic of *al-Mumtiʿ* is its adoption of an organizational style that moves from the simple to the complex throughout both the main body and sections of the text. In addition to the pedagogical motives, the author’s habit of composing works based on how the human mind works can be recalled as well.

وقدم الجمع والطرح على الضرب والقسمة. لانّ الجمع والطرح أسهل عملا منهما وأقرب الى الذهن.[62]

He gave precedence to addition and subtraction before division. For addition and subtraction are easier than multiplication and division, and closer to mind.

iv. Analytical form: Thanks to Ibn al-Hā’im’s remarkable analytical aptitude, *al-Mumtiʿ* is outstanding for its analytical form. All one has to do to see the truth of this statement is to study the section concerning calculations with algebraic expressions. In four basic arithmetic operations (viz., addition, subtraction, multiplication, and division), Ibn al-Hā’im classifies the terms of operation in each case, such as identical-variant, simple-complex, negative-positive, fraction, root, absolute number, type (x, x^2, x^3…), and so on. He then analyzes each of them with examples from all cases of combination. In this sense, one can suppose that *al-Mumtiʿ*, together with the author’s commentary of *al-Urjūza*, comprises the most comprehensive algebraic calculus ever produced within Islamic civilization.[63]

v. Explanation by examples: As a scholar who spent most of his life in teaching, Ibn al-Hā’im’s characteristic of effective teaching by providing a wealth of examples, as he does in all of his works, is also prevalent in *al-Mumtiʿ*.

vi. The field of arithmetical algebra: *al-Mumtiʿ* exhibits the characteristic of pure arithmetical algebra by presenting all of the given examples, operations, and proofs in terms of numbers, thereby entirely omitting geometrical examples and proofs. One can therefore infer that the author belongs to the ranks of arithmetical algebra as opposed to that of geometrical algebra. As a matter of fact, he provides

61 Ibid., 5a, 5b, 32b.

62 Ibid., 9b.

63 Ibid., 9b-29a; idem, *Sharḥ al-urjūzat al-yāsamīniyya fī al-jabr wa-al-muqābala*, 134–206.

the rationale for his position through his treatment of the two perspectives' controversial topics. This treatment, which is inclusive of the philosophical extensions of the debate's topics, secures the work's elevation to the status of an all-inclusive "encyclopedia" instead of its being no more than a "miscellany of algebraic rules" enriched with discussions of the philosophy of mathematics/algebra. For example, Ibn al-Hā'im begins his response after listing the opposing side's claims with the following sentences:

> ويترتب على ما ذكره: إن سلم سؤالان، أحدهما أن يقال فما فائدة ذكر مال المال ومال الكعب وما فوقهما في هذا العلم وثانيهما بطلان حصر المسائل الجبرية في الست المذكورة في النظم.[64]
>
> It follows from what he (Tāj al-Dīn Tabrīzī) spelled out: If two questions are accepted, first that says "What good is it to tell square of square, square of cube, and on, in this discipline?" and second "the refutation of the confinement of the algebraic equations to these six in the verse?"

After listing these arguments by singling out persons, he asks the above questions and, especially drawing from Ibn al-Bannā' al-Marrākushī, explains how and why there are quartic equations and those of higher degrees, as well as how the solution sets of these equations can be obtained, along with the accompanying justifications and mathematical examples.[65]

vii. Rhetorical style: Another aspect of *al-Mumti'* that needs to be mentioned here is the complete absence of any mathematical notion, to which many extant manuscripts contemporaneous with his work can attest, and the verbal expression of all mathematical operations. However, some of the elements he employed contradict this aspect.

> أن اهل الاصطلاح لهم في التعبير عن العدد في المسائل الجبرية طريقان: فمنهم من يذكره مطلقا من غير قيد، فيتميز بذلك عن غيره، كان يقال: ﴿ثلاثة وخمسة أشياء تعدل عشرة﴾، فتعلم أن الثلاثة والعشرة عددان وكذلك في الرسم بالهندي والغبار وتجعل لكل نوع علامة كالشين للأشياء والميم للأموال والكاف للكعوب وميمين لمال المال وهكذا، ...وهذا الطريق هو الذي سلكته في هذا الشرح غالبا لغرض الاختصار ومنهم من يميزه بتقييده بالدراهم او بالآحاد او بغير ذلك، فتقول ثلاثة دراهم او اربعة احاد او ثلاثة من العدد.[66]

64 İbnü'l-Hâim, *el-Mümti'*, vr. 55b.

65 Ibid., 57a-60a.

66 Ibid., 34b-35a.

> There are two methods for those proficient in terminology to express the number in algebraic equations: a) those who mention it (the number) as absolute without limits are among them (the proficient ones), they are duly distinguished, like saying "three plus five *shay'* is equal to ten." You know that three and ten are numbers, and the Indian or Persian figures as well. **For each kind you signify/make a notation: "shīn" for *shay'*s, "mīm" for squares (*māl*) "kāf" for cubes (*ka'b*) and two "mīm"s for square of square and likewise…This is the way I mainly followed in this commentary with the intent of abbreviation.** b) They are from them (the proficient ones) who reckons the number by counting in *dirham*, one, or other units, and tells "three *dirham*s or four ones or three from the numbers."

From these words, we can say that Ibn al-Hā'im expresses the equations in a manner similar to the algebraic symbolization in use today. However, no mathematical expression including figures is attested to in the manuscript. In order to delve further into this puzzle, there can be two reasons for this conundrum: (1) he might have made two sets of texts, one verbal and one numeral, in order to forestall any deviation within his work, as happens when words travel by way of mouth and during the course of further copying. To avoid this, he might have shared the first one openly and the second one only with those versed in the discipline and (2) he might not have followed what he endorsed at the beginning of the text, as can be seen in many other works by various authors. Nonetheless, one must concede that the attempt to convey mathematics in a symbolic language was present from the beginning of this discipline's development in the Muslim world and that significant results were achieved during each century because relying solely on the development of verbal expression to account for the serious upswing in calculations of knowns and unknowns, arithmetical operations with larger numbers, and multipage equation solutions is not possible. However, many historians of mathematics still adhere to formal and superficial analyses rather than substantially assessing the primary evidence at hand, and claim that mathematics in the Muslim world was largely devoid of notation even as late as the fifteenth century, except for the simple symbolizations of a few mathematicians.[67]

As Salih Zeki demonstrated, algebraic notation did not appear in a late stage, like seventeenth/thirteenth century (i.e., four centuries after the flourishing of mathematics in Islamic lands) due to a few western (Muslim) mathematicians.

67 For a few instances of these claims: Florian Cajori, *A History of Mathematical Notations* (London: 1928), I:84–85, 93; J. Mazur, *Enlightening Symbols: A Short History of Mathematical Notation and Its Hidden Powers* (Princeton: Princeton University Press, 2014); idem, *Matematik Sembollerinin Kısa Tarihi*, translated by Barış Gönülşen (Istanbul: İş Bankası Yayınları, 2016); Jan Cizmar, "The origins and development of mathematical notation (a historical outline)," *Quaderni di ricerca in didattica*, no. 9 (2000): 103–23; Stephen Wolfram, "Mathematical Notation: Past and Future," MathML International Conference, October 20, 2000.

Even though those works that used the earlier notation system are not at hand because of Arabic's structure and the reasons cited above, one can infer from later works that both the eastern and western Muslim mathematicians were aware of algebraic notation and gradually improved upon it.[68]

When Ibn al-Hā'im notes in *al-Mumti'* that Muslim mathematicians used symbolization even for classifying equation types, it is hard to believe that these same mathematicians could possibly be ignorant of the notation of equations.

> وإن كان متفقا عليه في المركبات عند اهل الصناعة وقد ضبطوا ترتيبها بقولك عجم كما اشار اليه بصدر البيت الأول فالعين للعدد والجيم للجذور والميم للمال اي فينفرد العدد في الضرب الأول والجذر في الثاني والمال في الثالث، وبالله التوفيق.[69]

> When the masters of the art concurred on the complex, they have registered it on the order of word ***'ajm***, as indicated in the middle of the first couplet, **'ayn for number (*'adad*), jīm for root (*jadhr*), and mīm for square (*māl*)**. That is, first the number is distinguished in multiplication, second the root, and third the square. By God, it shall succeed.

viii. "Warning/s": The work's last characteristic that needs to be noted is the author's fifteen headings of "warning" or "warnings" in order to stress the points that require the attention of whoever wants to acquire this body of knowledge, as well as his references to his other works in due course.

> ولعمري أنه إن لم يكن قد احكم الأعمال الخمسة التي تقدمت الإشارة اليها على ما ذكره الحساب، فلا يطمع في معرفة هذا الفن ولا يشم رائحته. فكم من المسألة تحير العاقل في تنصيفها الذي هو اصل الأعمال فضلا عن تجذير ها الذي هو اصعبها..... ولا بد من اتقان نحو وسيلتي والا فلا تطمع بأنك داخل.[70]

> On my life, if he has not mastered the five operations that I prioritized to point out along what the arithmeticians say, he neither would taste the learning of this science, nor smell its perfume. How many problems puzzled the savant in halving, which is the basis of operations, as well as taking square root, which is the hardest...It requires the mastery of my ***Wasīla*** and otherwise you won't get to marrow.

68 Salih Zeki, "Notation algébrique chez les Orientaux." Other studies research the algebraic symbolism in the mathematics developed in the Muslim world by thoroughly immersing themselves in primary sources. For instance: 'Abd al-Jawād al-Mahdī, "Ba'th al-rumūz fī al-ḥisāb wa-al-jabr wa taṭawwuruhā fī al-maghrib al-islāmī," in *al-'Aṣr al-dhahabī li-l-'ulūm fī al-buldān al-islāmiyya: al-Makhṭūṭāt al-'ilmiyyat al-maghāribiyya* (Algiers: Manshūrāt wizārat al-thaqāfat al-Jazā'ir, 2011), 25–36; Jeffrey A. Oaks, "Algebraic Symbolism in Medieval Arabic Algebra," *Philosophica* 87 (2012): 27–83.

69 Ibn al-Hā'im, *al-Mumti'*, 38a.

70 Ibid., 41b-42a.

ومن اراد الشجر في أعمال ذوات الأسماء والمنفصلات وسائر الجذور الصم فعليه بشرحي للياسمينية وأعلى منه ذلك كتابي المسمى بالمعونة وهو الذي لم ينسج على منواله ولم تسمع قريحة بمثاله وبالله التوفيق.[71]

Who would like to have elaboration on the operations of irrational polynomials and other irrational roots, has to have a look at my commentary on ***Yāsamīniyya***, and even better, my book called ***al-Maʿūna***, likeness of its form is not composed, and its grace unheard of. By God, it shall succeed.

c. The innovations of Ibn al-Hāʾim

Since Ibn al-Hāʾim, a prolific author with an inclination toward education, felt the need to mention his novel ideas in all of his works pertinent to the subject matter at hand, this section is titled "The innovations of Ibn al-Hāʾim" rather than of *al-Mumtiʿ* in order to be more comprehensive.

i. The most striking novelty of *al-Mumtiʿ*, as briefly stated in the "main features of the work" section above, is the provision of a terminological/conceptual glossary as a guide/companion for the reader/researcher. Even more, given its explication of conceptual relations with nuances attached to linguistic and philosophical debates, it is, in fact, more than a glossary – it is a textual edifice. In this sense, *al-Mumtiʿ* has to be distinguished from a similar work of algebra by this author, *Sharḥ al-Urjūzat al-yāsamīniyya fī al-jabr wa-al-muqābala*, for in the first work he had a free hand in devising its organizational and conceptual structure, thanks to its being a commentary on the author's own versified work, whereas he was constrained in the latter by the source text upon which he commented. However, the novel concepts of negative (*manfī*) and positive (*muthbat*) are present in both of Ibn al-Hāʾim's works of algebra.

إعلم أنهم يعبرون كثيرا عن المستثنى بالناقص وبالمنفي وعن المستثنى منه بالزائد وبالمثبت.... وأن الخارج من ضرب الزائد في الزائد ومن ضرب الناقص في الناقص، زائد ومن ضرب الزائد في الناقص، ناقص.[72]

Know that they mostly express what is deducted as minus and negative and what is deducted from as plus and positive… What comes out of the multiplication of the positive by positive and negative by negative is positive, of the multiplication of the positive with negative, negative.

71 Ibid., 42a.

72 Ibid., 18b-19a; idem, *Sharḥ al-urjūzat al-yāsamīniyya fī al-jabr wa-al-muqābala*, 143.

Given the author's manner of expression, he was not the one who specified the mathematical uses of "manfī" and "muthbat" to signify "negative" and "positive," respectively. However, as far as we know, the works that first employed these terms belong to him. In their study *Mathematicians, Astronomers, and Other Scholars of Islamic Civilizations and Their Works* (2003), Boris A. Rosenfeld and Ekmeleddin İhsanoğlu report that ʿAlī Qūshjī used these notions, that the terms themselves were in fact of Chinese origin and were transferred to Europe via Qūshjī and the Byzantine mathematicians, and employed in the mathematical terminology of Latin translations.[73] This interpretation needs to be revised, given that Ibn al-Hā'im used these very concepts approximately fifty years earlier than ʿAlī Qūshjī, and that his manner of use implies that their employment goes back even further.

ii. After the sentence that "what comes out of the multiplication of the positive by the positive and the negative by the negative is positive, of the multiplication of the positive with the negative, negative" quoted above, explaining why it is so in multiplication and then providing the justification, Ibn al-Hā'im mentions that it is hard to come by such a work that comprises these explanations, that is to say, his was a first on this subject.

...فقد ظهر لك السر في قولهم ضرب الزائد في الزائد، زائد وضرب الناقص في الناقص، زائد وضرب احدهما في الآخر، ناقص فافهم ذلك فانك لا تكاد تجده في غير هذا الشرح بهذا الشأن والله المستعان.[74]

The mystery in the words "The multiplication of the positive by positive is positive, and the multiplication of the negative by negative is positive, and the multiplication of the one with the other is negative." is revealed to you. Make sense of this. For you hardly find it in this manner somewhere other than this commentary. God is the Helper.

iii. It is also probably a novel aspect that this supporter of arithmetical algebra completely ignored geometry for the proof of mixed equations and instead provided numerical causes.[75] Nevertheless, with the caution of one who pursues truth, he acknowledges the need to consult geometrical proofs when the numerical causes are insufficient.

73 Rosenfeld and İhsanoğlu, *Mathematicians, Astronomers, and Other Scholars of Islamic Civilizations*, 286; ʿAlī Qūshjī, *Risālat al-Muḥammadiyya fī al-ḥisāb*, MS Ayasofya 2733, 136a.

74 Ibn al-Hā'im, *al-Mumti*ʿ, 21b.

75 Even though ʿAbd al-Jawād al-Mahdī, the editor of Ibn al-Hā'im's commentary on Yāsamīnī, suggests that the author borrowed the method from Ibn al-Bannā', there is no proof in the latter similar to Ibn Hā'im's. For further information, cf. Ibn al-Hā'im, *Sharḥ al-urjūzat al-yāsamīniyya fī al-jabr wa-al-muqābala*, 23; Aḥmad Salīm Saʿīdān, "Kitāb al-jabr wa-al-muqābala li- Ibn al-Bannā' al-Marrākushī," in *Tārīkh ʿilm al-jabr fī al-ʿālam al-ʿarabī*, II, 542–55.

وقد جرت عادة القوم أن يبينوا براهين هذه المسائل بالهندسة إما بالخطوط او بالسطوح ومعرفة ذلك تحقيقا تحوج الى معرفة أوقليدس. فرأيت ذلك بمقدمات عددية، من غير تعرّض لذكر خط او سطح، وإن كانت تلك المقدمات في نفسها مفتقرة الى البراهين الهندسية، وإنما أفعل ذلك تقريبا للمحصل وإحالة لبيان تلك المقدمات على أوقليدس او غيره من الكتب الهندسية.[76]

It has been customary to explain the proofs of these equations by geometry, that is, by lines and planes. Accurate knowledge of them requires the knowledge of **Euclid**. I decided on numerical premises, without going into the talk of line or plane. If the premises itself have been wanting in geometrical proofs, I would do it approximately to obtain the result, and leave the explanation of the premise to **Euclid** or other books of geometry.

iv. Again in mixed equations, for the first time Ibn al-Hā'im provides formulas to find "x" or "x^2" without relying on the "completion and reduction" method, which is based on converting the coefficient of the perfect square in the equation to one (1). The methods to obtain "x" and "x^2" are given below, respectively:

وهي المشار اليها ببقية الأبيات أن يبتغي في التوصل الى الجذر بما ذكر في النظم من غير جبر ولا حط.[77]

The thing referred to in the remainder of the couplets is the wish to extract root without completion and reduction.

مالان ونصف مال وعشرة أجذار تعدل مائة وخمسين. فاضرب عدة الأموال وهي إثنان ونصف في العدد يحصل ثلاثمائة وخمسة وسبعون فكأنّه العدد المفروض في الرابعة. فاعمل عملها المذكور في النظم اي زد مربع التنصيف وهو خمسة وعشرون على ثلاثمائة وخمسة وسبعين. يخرج أربع مائة وجذره عشرون فاطرح منه التنصيف، يبقى خمسة عشر، فاقسمها على الإثنين والنصف كما اشار اليه بقوله ﴿وفي الآخر اقسم ما لجذر يقابل على ما ضربت العد﴾ فيه. فيخرج ستة وهو الجذر المطلوب.[78]

Two squares plus half a square plus ten bases is equal to hundred and fifty. Multiply the coefficient of the square, which is two and a half, by the number and what comes out is three hundred and seventy-five. Do the operation mentioned in the verse as if it is the number assumed in the fourth, that is, add the square of halved, it is twenty-five, to three hundred and seventy-five, you'll get 400, and its root is twenty. Subtract the halved, i.e. five, from it, then remains fifteen. As the expression "consequently divide the balance of the root by the unknown multiplied by the number," signifies, divide it by two plus half, six comes out and it is the value that is unknown.

76 Ibn al-Hā'im, *al-Mumti'*, 42b; idem, *Sharḥ al-urjūzat al-yāsamīniyya fī al-jabr wa-al-muqābala*, 79.

77 Ibn al-Hā'im, *al-Mumti'*, 51b.

78 Ibid., 51b-52a.

$$2x^2 + \frac{x^2}{2} + 10x = 150 \qquad x = \frac{\sqrt{\left(2+\frac{1}{2}\right).150 + \left(\frac{10}{2}\right)^2} - \frac{10}{2}}{2+\frac{1}{2}} = \frac{\sqrt{375+25} - 5}{\frac{5}{2}} = \frac{15}{\frac{5}{2}} = 6 \rightarrow x = 6$$

إنك إذا اردت الخروج ابتداء الى المال حيث كان المفروض في المركبة اقل من مال او أكثر من مال من غير جبر ولا حط، فلك ذلك....[79]

When you want to obtain the square without making use of completion and reduction, in case the coefficient of the square is smaller or bigger than the square, it is for you...

ففي مالين ونصف مال وعشرة أجذار تعدل مائة وخمسين اضرب العدد في الإثنين والنصف عدة الأموال ثم ربع الحاصل يحصل مائة وأربعون الفا وستمائة وخمسة وعشرون فاحفظه، ثم زد على مضروب العدد في عدة الأموال وهو ثلاثمائة وخمسة وسبعون، نصف مربع عدة الأجذار وهو خمسون يحصل اربع مائة وخمسة وعشرون فاحفظه، ثم اطرح المحفوظ الأول من مربع المحفوظ الثاني وهو مائة وثمانون الفا وستمائة وخمسة وعشرون، يبق أربعون الفا فاطرح جذره وهو مائتان من المحفوظ الثاني واقسم الباقي وهو مائتان وخمسة وعشرون على مربع عدة الأموال وهو ستة وربع يحصل ستة وثلاثون وهو المال المطلوب.[80]

In (the equation of) "two squares plus half a square plus ten unknowns is equal to hundred and fifty," multiply the number by two and a half, then take the square of the result, hundred and forty thousand six hundred and twenty-five comes out, keep it. Then add half of the square of the coefficient of unknown, i.e. fifty, to the multiplication of the number by the coefficient of squares, i.e. three hundred and seventy-five, four hundred and twenty-five comes out, keep it. Then subtract the first one kept, from the square of the second, i.e. hundred and eighty thousand six hundred and twenty-five, forty thousand remains. Subtract the root of it, i.e. two hundred, from the second one kept, divide the remainder, i.e. two hundred and twenty-five, by the square of the number of squares, i.e. six plus quarter, thirty six comes out, that is the value of square.

$$2x^2 + \frac{x^2}{2} + 10x = 150 \longrightarrow$$

$$x^2 = \frac{\left(2+\frac{1}{2}\right).150 + \frac{10^2}{2} - \sqrt{\left(\left(2+\frac{1}{2}\right).150 + \frac{10^2}{2}\right)^2 - \left(\left(2+\frac{1}{2}\right).150\right)^2}}{\left(2+\frac{1}{2}\right)^2} =$$

$$\frac{375 + 50 - \sqrt{180625 - 140625}}{6+\frac{1}{4}} = \frac{425 - \sqrt{40000}}{6+\frac{1}{4}} = \frac{425 - 200}{6+\frac{1}{4}} = \frac{225}{6+\frac{1}{4}} = \mathbf{36}$$

79 Ibid., 53a.

80 Ibid., 53a-54b.

v. *Al-Mumti'* is among the few studies that extensively discuss whether or not the number of equations can be limited under a separate heading. Thus, Ibn al-Hā'im first sets forth the names of those who argue that it can, these mathematicians' pertinent claims, and their conclusions in reference with the names of their works. He then tries to refute them one by one together with borrowing from the works of his predecessors. As a brief example of the opposing claims:

زعم تاج الدين التبريزي أن المقادير التي تدور عليها المسائل الجبرية منحصرة ضرورة في هذا العدد والجذور والأموال والكعوب وبنى على ذلك أن المسائل الدائرة على هذه الأربعة، خمس وعشرون، وعزى ذلك الى عمر الخيام، قال: ﴿ولا يمكن أن يقع أكثر من هذه المقادير الأربعة، لان مال المال لا يقع الا في المقادير ووقوعه فيها محال. اذ المقادير ذو بعد واحد وهو الجذر والضلع، وذو بعدين وهو المال والسطح، ذو الأبعاد الثلاثة وهو الكعب والجسم ولا بعد آخر. فيقع فيها مال المال فضلا عما فوقه. واذا قيل : ﴿مال المال﴾، فانما يقال ذلك لعدد أجزائها عند المساحة لا لذواتها ممسوحة واذا لم يقع مال المال وما فوقه، فتكون المقادير منحصرة ضرورة في الأربعة. »[81]

Tāj al-Dīn al-Tabrīzī claimed that the values/expressions employed by algebraic equations were necessarily limited by number, root, square, and cube. On top of it, he proposed that there are 25 equations that use these four and attributed it to **'Umar Khayyām** and said: "It is not possible to be more than these four terms. For square of square only exists in value, and it does not exist. Then, if the expression is one-dimensional, it is variable and root of a third degree and higher, if two-dimensional, square and plane, and if third-dimensional, cube and object, and there are no other dimensions. Square of square is present there together with those of higher degrees as a supplement. When said square of square in value, it is due to the number of its parts by measure, not the primary forms that was measured. There is a difference between the two: square of square is present in value neither in essence nor in accident. Since square of square and higher did not exist, the algebraic expressions are necessarily limited to four.

vi. After having introduced many equations, varying in type and form, and explained each one's solution methods and proofs, the author conveys what should be done before applying the given information to the problem encountered. Thus, not every equation or problem is a "question," for it has to display certain minimum conditions in order to qualify for that particular designation. In case these conditions are ignored, especially by people occupied with algebra, it is probable that they might tire themselves in vain or even make fools of themselves in order to obtain results when confronted with non-"question" problems. Ibn al-Hā'im shows his talent at using all of the disciplines in which he was versed together when necessary and puts forth the *conditio per quam* for the equation by borrowing from logic for the very first time in a systematic manner.

81 Ibid., 54b-55a.

إن كل مسألة ترد عليك ويطلب منك جوابها فلإمكان الوصول الى السؤال ثلاثة شروط:

أحدها أن تكون المسألة في نفسها ممكنة والا فلا جواب لها فلا يبتغي.

الشرط الثاني؛ أن يكون في المسألة ثلاث معلومات فصاعدا. والمعلوم ضربان: معلوم الكمية كعشرة ويلحق به نحو جذر عشرة ومعلوم الكيفية كزيادة نصف العدد عليه او نقصانه منه او ضربه في معلوم او قسمته على معلوم او تربيعه او غير ذلك.

الشرط الثالث؛ أن يكون بين المعلوم المفروض وبين المجهول المطلوب ارتباط ووصلة بحيث يتوصل منه اليه.[82]

There are three conditions for every equation/problem, which was posed and a result was asked, to be able to reach the (status of) "question":

One is the possibility of the equation's being in itself, otherwise neither there will be an answer nor it will be asked.

Second condition, it is the presence of three or more knowns in the equation. **Known is of two types**:

1. The **knowledge of quantity** like "ten," so that the "root then" and the like are also of this type.

2. The **knowledge of quality** like the addition/subtraction of the half of the number to/from it, or multiplication/division by the known, or taking its square, or likewise.

Third condition, it is the presence of a relation or link between the assumed known and the requested unknown that allows to reach the latter from the former.

vii. Finally, the provision of information, even if too brief, on the practical counterparts of the theoretical knowledge given throughout the text that would or could respond to human needs is notable for giving an idea about the extent to which algebra could actually facilitate human life.

وقد لا يصرح في السؤال بشيء من هذه الأقسام غير أنه يذكر فيه ما يرجع اليها كأكثر مسائل البيع والشراء والإجارة والمرابحة وكمسائل البريد والتلاقي ومسائل الليل والحياض والطيور وكغالب مسائل الوصايا والأقرار بالدين وغير ذلك من المسائل الدورية كالهبة والعتق والمحاباة في البيع والشراء والسلم والاقالة والضمان والشفعة والصداق والخلع والكتابة والجناية ومسائل الإنتهاب والتركات المجهولة.[83]

He might have not explained of these parts in the question, however, he mentions there that turns to donation, manumission, favoritism in purchase-sale, sale on cash, termination of sale, surety, the right of neighborliness, contract of mar-

82 Ibid., 60b-61b.

83 Ibid., 61b-62a.

riage, dismissal, penmanship, commission of crime, and other periodical/cyclical equations/problems, and the problems of plunder and unknown inheritance, in the most of the problems of purchase, sale, rent, and profit, in the equations of mail, encounter, night, menstruation, and birds, and in the majority of the equations of will and testament in religion.

Conclusion

i. *Al-Mumti'* synthesizes the methods and principles of the algebra developed and pursued in the eastern and western lands of the Muslim world (*al-Mashriq* and *al-Maghrib*) by taking into account the contemporaneous reserve of knowledge and draws upon multiple sources to realize it. This aspect can be attested to not only in the works of algebra, but also in the author's other works. At this juncture, if the primary sources of Ibn al-Hā'im are to be recalled, it is al-Karajī (410/1019) in the eastern Muslim world and Ibn al-Bannā' al-Marrākushī (721/1321) in the western Muslim world.

ii. The author's main method of composition, that of organizing both the outline and the subsections from easy to complex and presenting all of the concepts current in that discipline together with definitions and explications, can also be attested to in *al-Mumti'*.

iii. While the work draws upon multiple sources, they are combined in such a way that they do not contradict the system mentioned in the previous article.

iv. *Al-Mumti'* is not content with a single method in either the arithmetical operations (including extracting roots with algebraic expressions) or in the solution of all types of equations. Rather, it proves the flexible form of algebra by viewing the problem from different angles and figuring out alternative methods of solution. Thus it is one of works at the peak of expanding the discipline's boundaries through the arithmetization of algebra that began with al-Karajī's school.

v. Even though the work seems quite similar to another of his works on algebra, *Sharḥ al-Urjūzat al-yāsamīniyya fī al-jabr wa-al-muqābala*, even to the extent of encountering the same explanations and numerical examples in many places, one can say that it is one step ahead as regards (1) its ability to present what the author wanted to set forth concerning algebra without being subject to any limitations by the merit of *al-Mumti'* being the commentary of his own versified work and (2) the composition of *al-Mumti'* two decades after *Sharḥ al-Urjūza* so that that it could represent his learning in an advanced form.

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Appendix 1: Glossary of Algebraic Terms

1. *al-Shay'*(x) /الشَيْء : The variable in an algebraic equation whose value is unknown and is subsequently made known through the known values via certain operations.
2. *al-Māl*(x^2) /المَال : The square of the variable obtained by the multiplication of "*al-shayy*" by itself.
3. *al-Ka'b*/(x^3) /الكَعْب : The cube obtained by multiplying the variable by its square.
4. *al-Jadhr*/الجَذْر : Square root. Since the root of the equation is "x," it stood for the variable together with "*al-shay'*" in the early period of algebra. Later on it came to mean the square root of a constant.
5. *al-Majdhūr*/المجذور : That which has a square root. It is perfect square number.
6. *al-Ḍil'*/ الضِّلْع : The root of a number from the third or higher degree.
7. *al-Murabba'*/المربّع : The result of taking the square of any value, namely, multiplying it by itself.
8. *al-Muka'ab*/المكعّب : The result of multiplying any value three times by itself.
9. *al-Jabr*/الجبر : The word *algebra* marks the two methods/operations used in equations: (1) the completion of the value of the negative expression on the left/right side or both sides, so that the equation will be positive and (2) when the coefficient of the square (x^2) is smaller than 1, it is the method used to convert the coefficient to 1 in order to reduce the equation to one of the standard equation types.
10. *al-Muqābala*/ المقابلة : Balancing, or juxtaposing similar terms by comparing the expressions on the left and right sides of the equation.
11. *al-Zā'id* and *al-Muthbat*/ الزائد والمثبت: Plus and positive, respectively.
12. *al-Nāqiṣ* and *al-Manfī*/ الناقص والمنفي : Minus and negative, respectively.
13. *al-Jam'*/الجمع : Addition.
14. *al-Ṭarḥ*/ الطرح : Subtraction.
15. *al-Ḍarb*/الضرب : Multiplication.
16. *al-Qisma*/ القسمة : Division.
17. *al-Munṭaq*/ المنطق : All forms of numbers in the set of rational numbers.
18. *al-Aṣamm*/الأصم : All forms of numbers in the set of irrational numbers.
19. *al-Majhūl*/ المجهول : Variable.

Appendix 2: Mathematical Analysis[84]

1. Basic Terms of Algebra and Their Distinctions[85]

- Unknown terms: Primary ones: $x,\ x^2,\ x^3$ *and* Secondary ones: $x^4,\ x^5,\ x^6 \ldots \infty$
- $x = 3 \Rightarrow x = shayy'$ *and* $x.x = 9 \Rightarrow x = root$ (*with respect to being root*)
- $x = \sqrt{5} \Rightarrow x \in I_r,\quad x^2 = \left(\sqrt{5}\right)^2\ x^2 = 5 \Rightarrow x^2 \in Q$
- $\forall\, a, b, c \in R\backslash\{0\}$ *and* $a.b = c \Rightarrow \mathrm{c} = musaṭṭiḥ/saṭḥ/basīṭ$
- $\forall\, a, b \in R\backslash\{0\}$ *and* $a.a = b \Rightarrow \mathrm{b} = square/murabba^{ʿ}/majdhūr = musaṭṭiḥ/$ $saṭḥ/basīṭ \Rightarrow \{square = murabba^{ʿ} = unknown\} \subset \{musaṭṭiḥ = satıh = basīṭ\}$
- $root \subset ḍilʿ$
- According to one view: $mukaʿab = kaʿb \Rightarrow 2.2^2 = 8,\ 8 = mukaʿab$ *and* $2 =$ $ḍilʿ$
- According to another view: $mukaʿab \neq kaʿb \Rightarrow 2.2^2 = 8,\ 8 =$ $mukaʿab$ *and* $2 = kaʿb$

  (According to the author this view is preferable, because "$ḍilʿ \supset$ root, etc.")

2. Addition and Subtraction with Algebraic Expressions[86]

- $\forall\, \boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{R}\backslash\{\boldsymbol{0}\}$ ***and*** $\boldsymbol{n}, \boldsymbol{m} \in \boldsymbol{N} \Rightarrow \boldsymbol{ax^n} + \boldsymbol{bx^n} = (\boldsymbol{a} + \boldsymbol{b})\boldsymbol{x^n},\ \boldsymbol{ax^m} + \boldsymbol{bx^m} =$ $(\boldsymbol{a} + \boldsymbol{b})\boldsymbol{x^m},\ \cdots$
- $\forall \boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{R}\backslash\{\boldsymbol{0}\},\ \boldsymbol{a} > \boldsymbol{b}$ ***and*** $\boldsymbol{n}, \boldsymbol{m} \in \boldsymbol{N} \Rightarrow \boldsymbol{ax^n} - \boldsymbol{bx^n} = (\boldsymbol{a} - \boldsymbol{b})\boldsymbol{x^n},\ \boldsymbol{ax^m} -$ $\boldsymbol{bx^m} = (\boldsymbol{a} - \boldsymbol{b})\boldsymbol{x^m}\ \ldots$

84 Since all of the operations in a voluminous book of algebra exceed the limits of this article, and since the main aim is to provide a general frame for *al-Mumti*ʿ, only the essential elements of the book will be dealt with herein.

85 Ibn al-Hāʾim, *al-Mumti*ʿ, 1b-9b.

86 Ibid., 9b-14b.

- $(15x^2 + 15x^3) - (8x^2 + 7x^3) = 7x^2 + 8x^3$
- $(5x^3 - 3x) - (4x^2 - 2) = (5x^3 - 3x + \mathbf{2} + \mathbf{3x}) - (4x^2 - 2 + \mathbf{2} + \mathbf{3x}) =$ $(5x^3 + 2) - (4x^2 + 3x)$
- $10x^2 - 10x = 30x^2 - 100 \Rightarrow 10x^2 - 10x + \mathbf{10x} + \mathbf{100} = 30x^2 - 100 + \mathbf{10x} + \mathbf{100} \Rightarrow 10x^2 + 100 = 30x^2 + 10x \Rightarrow 100 = 20x^2 + 10x$

3. Multiplication and Division by Algebraic Expressions[87]

- $\forall\, \boldsymbol{a, b, c, d} \in \boldsymbol{R}\backslash\{\mathbf{0}\}$ ***and*** $\boldsymbol{n, m} \in \boldsymbol{N} \Rightarrow (\boldsymbol{a} + \boldsymbol{bx^n}).(\boldsymbol{cx^n} + \boldsymbol{dx^m}) = \boldsymbol{acx^n} + \boldsymbol{adx^m} + \boldsymbol{bcx^{2n}} + \boldsymbol{bdx^{n+m}}$
- $\forall\, \boldsymbol{a, b} \in \boldsymbol{R}\backslash\{\mathbf{0}\}$ ***and*** $\boldsymbol{n, m, p} \in \boldsymbol{N} \Rightarrow \sqrt{\sqrt{\boldsymbol{a^m}}}.\boldsymbol{b^p x^n} = \sqrt{\sqrt{\boldsymbol{a^m}.(\boldsymbol{b^p x^n})^4}} = \sqrt{\sqrt{\boldsymbol{a^m}.\boldsymbol{b^{4p}}.\boldsymbol{x^{4n}}}}$
- $\forall\, \boldsymbol{a, b, c, d} \in \boldsymbol{R}\backslash\{\mathbf{0}\}$ ***and*** $\boldsymbol{n, m} \in \boldsymbol{N} \Rightarrow (\boldsymbol{a} - \boldsymbol{bx^n}).(\boldsymbol{cx^n} - \boldsymbol{dx^m}) = \boldsymbol{acx^n} - \boldsymbol{adx^m} - \boldsymbol{bcx^{2n}} + \boldsymbol{bdx^{n+m}}$
- $+.+= +, \quad -.-= +, \quad +.-= -, \quad -.+= -$
- $\forall\, \boldsymbol{a, b, c} \in \boldsymbol{R}\backslash\{\mathbf{0}\}$ ***and*** $\boldsymbol{n, m} \in \boldsymbol{N} \rightarrow \frac{\boldsymbol{ax^n}}{\boldsymbol{bx^m}} = \boldsymbol{c} \Rightarrow \boldsymbol{ax^{n-m}} = \boldsymbol{bc}$ ***and*** $\boldsymbol{bx^m}.\boldsymbol{c} = \boldsymbol{ax^n}$
- $\frac{\boldsymbol{x^2}}{\boldsymbol{x}} = \frac{\boldsymbol{x^3}}{\boldsymbol{x^2}} = \frac{\boldsymbol{x^4}}{\boldsymbol{x^3}} = \frac{\boldsymbol{x^5}}{\boldsymbol{x^4}} = \cdots = \boldsymbol{x}$ ***and*** $\frac{\boldsymbol{x^3}}{\boldsymbol{x}} = \frac{\boldsymbol{x^4}}{\boldsymbol{x^2}} = \frac{\boldsymbol{x^5}}{\boldsymbol{x^3}} = \frac{\boldsymbol{x^6}}{\boldsymbol{x^4}} = \cdots = \boldsymbol{x^2}$
  - $\left(\frac{1}{3} + \frac{1}{4}\right).\left(2x^3 + \frac{x^3}{2}\right) = x^3 + \frac{x^3}{3} + \frac{x^3}{8}$
  - $(4x + 3x^2 + 5x^3).(4 + 3x + 5x^2 + 6x^3) = 16x + 12x^2 + 20x^3 + 24x^4 + 12x^2 + 9x^3 + 15x^4 + 18x^5 + 20x^3 + 15x^4 + 25x^5 + 30x^6 = 16x + 24x^4 + 49x^3 + 54x^4 + 43x^5 + 30x^6$

87 Ibid., 15a-29a.

- $\sqrt{\sqrt{3}}.2x^2 = \sqrt{\sqrt{3.16x^8}} = \sqrt{\sqrt{48x^8}}$
- $[(10+10x)-(x^2+x^3)].[(20+15x)-(3x^2+4x^3)] = 200+150x-30x^2-40x^3+200x+150x^2-30x^3-40x^4-20x^2-15x^3+3x^4+4x^5-20x^3-15x^4+3x^5+4x^6 = 200+350x+100x^2-105x^3-52x^4+7x^5+4x^6$
- $999.999\times999.999 = (1.000.000-1).(1.000.000-1)$
- $\frac{10x}{x+1}.\frac{20}{x} = \frac{200x}{x(x+1)} = \frac{200x}{x^2+x}\ or\ \frac{10x}{x+1}.\frac{20}{x} = \frac{200}{x+1}$
- $\frac{10x+5x^2}{x+1}.\frac{20+6x^2}{x+2} = \frac{200x+60x^3+100x^2+30x^4}{2+3x+x^2}$
- $\frac{\frac{10}{x}-x}{\frac{3}{x}} \rightarrow \frac{10}{x}.\frac{x}{3} = \mathbf{3}+\frac{\mathbf{1}}{\mathbf{3}} \Rightarrow -x.\frac{x}{3} = -\frac{x^2}{3} \Rightarrow \frac{\frac{10}{x}-x}{\frac{3}{x}} = 3+\frac{1}{3}-\frac{x^2}{3}$
- $\frac{\frac{10}{x^2}-x}{\frac{3x}{x^2-3}} = \frac{\frac{10}{x^2}}{\frac{3x}{x^2-3}} - \frac{x}{\frac{3x}{x^2-3}} = \left[\frac{10}{x^2}.\left(\frac{x^2}{3x}-\frac{3}{3x}\right)\right] - \left[x.\left(\frac{x^2}{3x}-\frac{3}{3x}\right)\right] = \left(\frac{10}{3x}-\frac{10}{x^3}\right) - \left(\frac{x^2}{3}-1\right) = 1+\frac{3\frac{1}{3}}{x}-\frac{x^2}{3}-\frac{10}{x^3}$

4. Taking the Root of Polynomial[88]

- $\forall\ \boldsymbol{a,b,c,d,e} \in \boldsymbol{R}\backslash\{\boldsymbol{0}\}$ ***and m, n, p, q consecutive positive integers*** $\Rightarrow$

$$\sqrt{\boldsymbol{a+bx^m+cx^n+dx^p+ex^q}} = \frac{\boldsymbol{cx^n} - \left(\frac{\frac{\boldsymbol{dx^p}}{\sqrt{\boldsymbol{ex^q}}}}{\boldsymbol{2}}\right)^{\boldsymbol{2}}}{\frac{\sqrt{\boldsymbol{ex^q}}}{\boldsymbol{2}}} + \frac{\frac{\boldsymbol{dx^p}}{\sqrt{\boldsymbol{ex^q}}}}{\boldsymbol{2}} + \sqrt{\boldsymbol{ex^q}}$$

88 Ibid., 29a-30b.

$$\bullet\forall \{\boldsymbol{a,b,c,d,e,f,g}\} \in \boldsymbol{Q}\backslash\{\boldsymbol{0}\}, \sqrt{\boldsymbol{a}}\ \boldsymbol{and}\ \sqrt{\boldsymbol{gx^l}} \in \boldsymbol{Q}\ \boldsymbol{and}\ \{\boldsymbol{m,n,p,q,k,l}\}\ \boldsymbol{consecutive} \in \boldsymbol{N} \Rightarrow \sqrt{\boldsymbol{a+bx^m+cx^n+dx^p+ex^q+fx^k+gx^l}}$$

$$= \frac{\dfrac{dx^p - \left(\dfrac{\dfrac{\boldsymbol{ex^q} - \left(\dfrac{\frac{\boldsymbol{fx^k}}{\sqrt{\boldsymbol{gx^l}}}}{2}\right)^2}{\sqrt{\boldsymbol{gx^l}}}}{2} . \dfrac{\frac{\boldsymbol{fx^k}}{\sqrt{\boldsymbol{gx^l}}}}{2} . 2\right)}{\sqrt{\boldsymbol{gx^l}}}}{2} + \frac{\dfrac{\boldsymbol{ex^q} - \left(\dfrac{\frac{\boldsymbol{fx^k}}{\sqrt{\boldsymbol{gx^l}}}}{2}\right)^2}{\sqrt{\boldsymbol{gx^l}}}}{2} + \frac{\frac{\boldsymbol{fx^k}}{\sqrt{\boldsymbol{gx^l}}}}{2} + \sqrt{\boldsymbol{gx^l}}$$

➢ $\sqrt{9+12x+10x^2+4x^3+x^4} =$

$$\frac{\dfrac{10x^2 - \left(\dfrac{\frac{4x^3}{\sqrt{x^4}}}{2}\right)^2}{\sqrt{x^4}}}{2} + \frac{\frac{4x^3}{\sqrt{x^4}}}{2} + \sqrt{x^4} = \frac{\frac{10x^2-4x^2}{\sqrt{x^4}}}{2} + 2x + x^2 = \boldsymbol{3+2x+x^2}$$

- $\sqrt{4+8x+12x^2+16x^3+12x^4+8x^5+4x^6} = \frac{\frac{16x^3-\left(\frac{\frac{12x^4-\left(\frac{8x^5}{\frac{\sqrt{4x^6}}{2}}\right)^2}{\sqrt{4x^6}}}{2}.\frac{8x^5}{\frac{\sqrt{4x^6}}{2}}.2\right)}{\sqrt{4x^6}}}{2} +$

$\frac{\frac{12x^4-\left(\frac{8x^5}{\frac{\sqrt{4x^6}}{2}}\right)^2}{\sqrt{4x^6}}}{2} + \frac{\frac{8x^5}{\sqrt{4x^6}}}{2} + \sqrt{4x^6} = \frac{\frac{16x^3-\left(\frac{\frac{12x^4-4x^4}{\sqrt{4x^6}}}{2}.2x^2.2\right)}{\sqrt{4x^6}}}{2} + \frac{\frac{12x^4-4x^4}{\sqrt{4x^6}}}{2} + \frac{4x^2}{2} + 2x^3 =$

$\frac{\frac{16x^3-(2x.2x^2.2)}{2x^3}}{2} + \frac{\frac{8x^4}{2x^3}}{2} + 2x^2 + 2x^3 = \mathbf{2+2x+2x^2+2x^3}$

5. Indeterminate Equations/Induction[89]

- $assuming\ that\ \ x^2+4x=y^2\ and\ y=2x\ \Rightarrow\ x^2+4x=(2x)^2\ \Rightarrow x^2+4x=$ $4x^2\ \rightarrow\ \ 4x=3x^2\ \ \Rightarrow\ \ x=1+\frac{1}{3}\ and\ x^2=1+\frac{7}{9}\Rightarrow x^2+4x=1+\frac{7}{9}+$ $4.\left(1+\frac{1}{3}\right)=7+\frac{1}{9}=y^2\ \ \Rightarrow\ \ \boldsymbol{y=2+\frac{2}{3}}$
- $assuming\ that\ x^2+4x=y^2\ and\ y=x\ the\ equation\ will\ be\ void.$
- $assuming\ that\ x^2+4x=y^2\ and\ y=x+\frac{x}{2}\ \Rightarrow\ \ x^2+4x=\left(x+\frac{x}{2}\right)^2\Rightarrow\ x^2+$ $4x=\frac{9x^2}{4}\ \Rightarrow 4x=\frac{5x^2}{4}\ \Rightarrow x=3+\frac{1}{5}\ and\ x^2=10+\frac{1}{5}+\frac{1}{5}.\frac{1}{5}\ \ \Rightarrow x^2+4x=10+$ $\frac{1}{5}+\frac{1}{5}.\frac{1}{5}+4.\left(3+\frac{1}{5}\right)=23+\frac{1}{5}.\frac{1}{5}=y^2\ and\ y=\mathbf{4+\frac{4}{5}}$

89 Ibid., 31a-32a.

- $assuming\ that\ x^2 + 4x = y^2\ and\ y = x - 1 \Rightarrow x^2 + 4x = (x-1)^2 \Rightarrow x^2 + 4x = x^2 - 2x + 1 \Rightarrow 6x = 1 \rightarrow x = \frac{1}{6}\ \ and\ \ x^2 = \frac{1}{4}.\frac{1}{9} \Rightarrow x^2 + 4x = \frac{1}{4}.\frac{1}{9} + 4.\frac{1}{6} = \frac{2}{3} + \frac{1}{4}.\frac{1}{9} = y^2\ \ and\ \ \boldsymbol{y = \frac{5}{6}}$

6. Six Algebraic Equations[90]

- **Simple equations**
  - $\forall\, \boldsymbol{a, b} \in \boldsymbol{R}\backslash\{\boldsymbol{0}\} \Rightarrow \boldsymbol{ax^2 = bx} \Rightarrow \boldsymbol{x = b/a}$
  - $2x^2 + \frac{x^2}{4} = 9x \Rightarrow \frac{9}{\left(2+\frac{1}{4}\right)} = 4 \Rightarrow x = 4, x^2 = 16, \left(2 + \frac{1}{4}\right)x^2 = 36 = 9x$
  - $\forall\, \boldsymbol{a, c} \in \boldsymbol{R}\backslash\{\boldsymbol{0}\} \Rightarrow \boldsymbol{ax^2 = c} \Rightarrow \boldsymbol{x = \sqrt{c/a}}$
  - $\frac{x^2}{3} + \frac{x^2}{4} = 21 \Rightarrow \frac{21}{\frac{1}{3}+\frac{1}{4}} = 36 \Rightarrow x^2 = 36 \Rightarrow \frac{36}{3} + \frac{36}{4} = 21$
  - $\forall\, \boldsymbol{c, b} \in \boldsymbol{R}\backslash\{\boldsymbol{0}\} \Rightarrow \boldsymbol{bx = c} \Rightarrow \boldsymbol{x = c/b}$
  - $3x + \frac{x}{6} + \frac{x}{9} = 2 + \frac{5}{9} \Rightarrow \frac{2+\frac{5}{9}}{3+\frac{1}{6}+\frac{1}{9}} = 6 + \frac{40}{59} = x \Rightarrow 3\left(6 + \frac{40}{59}\right) + \frac{6+\frac{40}{59}}{6} + \frac{6+\frac{40}{59}}{9} = 2 + \frac{5}{9}$
- **Mixed equations**
  - $\forall\, \boldsymbol{a, b, c} \in \boldsymbol{R}\backslash\{\boldsymbol{0}\} \Rightarrow \boldsymbol{ax^2 + bx = c} \Rightarrow \boldsymbol{x = \sqrt{\left(\frac{b}{2}\right)^2 + c} - \frac{b}{2}}$
  - $x^2 + 7x = 8 \Rightarrow x = \sqrt{\left(\frac{7}{2}\right)^2 + 8} - \frac{7}{2} = \sqrt{12 + \frac{1}{4} + 8} - 3 + \frac{1}{2} = \sqrt{20 + \frac{1}{4}} - 3 + \frac{1}{2} = 4 + \frac{1}{2} - \left(3 + \frac{1}{2}\right) = 1 \Rightarrow x^2 = 1\ and\ 1 + 7 = 8$
  - $\forall\, \boldsymbol{a, b, c} \in \boldsymbol{R}\backslash\{\boldsymbol{0}\} \Rightarrow \boldsymbol{ax^2 + c = bx} \Rightarrow \boldsymbol{c = \left(\frac{b}{2}\right)^2\ and\ c < \left(\frac{b}{2}\right)^2} \Rightarrow \boldsymbol{x \neq \varnothing}$

    $\boldsymbol{and\ \ x = \frac{b}{2} \mp \sqrt{\left(\frac{b}{2}\right)^2 - c}\ \ \ and\ \ \ c > \left(\frac{b}{2}\right)^2} \Rightarrow \boldsymbol{x = \varnothing}$
  - $x^2 + 25 = 10x \Rightarrow x = \frac{10}{2} \mp \sqrt{\left(\frac{10}{2}\right)^2 - 25} = 5 \Rightarrow x = 5\ and\ x^2 = 25$

90 Ibid., 32a-44b.

➢ $x^2+6+\frac{7}{8}+\frac{1}{2}.\frac{1}{8}=10x \quad \Rightarrow \quad x=\frac{10}{2}\mp\sqrt{\left(\frac{10}{2}\right)^2-\left(6+\frac{7}{8}+\frac{1}{2}.\frac{1}{8}\right)}=5\mp$ $\sqrt{18+\frac{1}{2}.\frac{1}{8}}=5\mp\left(4+\frac{1}{4}\right) \quad \Rightarrow \quad x_1=\frac{3}{4},\ x_1^2=\frac{1}{2}+\frac{1}{2}.\frac{1}{8} \quad \Rightarrow \quad 10x_1=7+$ $\frac{1}{2}$ *and* $x_2=9+\frac{1}{4} \quad \Rightarrow \quad x_2^2=85+\frac{1}{2}+\frac{1}{2}.\frac{1}{8} \quad \Rightarrow \quad 10x_2=92+\frac{1}{2}$

➢ $x^2+30=10x \rightarrow x=\frac{10}{2}\mp\sqrt{\left(\frac{10}{2}\right)^2-30}=5\mp\sqrt{25-30}=5\mp\sqrt{-5}=\varnothing$

➢ $\forall\, \boldsymbol{a,b,c} \in \boldsymbol{R}\backslash\{\boldsymbol{0}\} \ \Rightarrow \ \boldsymbol{bx+c=ax^2} \ \Rightarrow \ \boldsymbol{x}=\sqrt{\left(\frac{\boldsymbol{b}}{\boldsymbol{2}}\right)^2+\boldsymbol{c}}+\frac{\boldsymbol{b}}{\boldsymbol{2}}$

➢ $x^2=x\left(1+\frac{5}{6}\right)+1+\frac{1}{5}+\frac{4}{5}.\frac{1}{5} \quad \Rightarrow \quad x=\sqrt{\left(\frac{1+\frac{5}{6}}{2}\right)^2+1+\frac{1}{5}+\frac{4}{5}.\frac{1}{5}}+\frac{1+\frac{5}{6}}{2}=$ $\sqrt{\left(\frac{2}{3}+\frac{1}{4}\right)^2+1+\frac{1}{5}+\frac{4}{5}.\frac{1}{5}}+\frac{2}{3}+\frac{1}{4}=\sqrt{2+\frac{1}{5}+\frac{1}{5}.\frac{1}{8}.\frac{1}{9}.\frac{1}{10}}+\frac{2}{3}+\frac{1}{4}=1+\frac{1}{4}+\frac{1}{5}+$ $\frac{1}{3}.\frac{1}{10}+\frac{2}{3}+\frac{1}{4}=2+\frac{2}{5}=x \quad \Rightarrow \quad x^2=5+\frac{3}{5}+\frac{4}{5}.\frac{1}{5}$

- **The cause of the first type of mixed equation**

➢ $\forall\, \boldsymbol{a,b} \in \boldsymbol{R}\backslash\{\boldsymbol{0}\} \Rightarrow \boldsymbol{a}=\frac{\boldsymbol{a}}{\boldsymbol{2}}+\frac{\boldsymbol{a}}{\boldsymbol{2}} \quad \Rightarrow \quad (\boldsymbol{a+b}).\boldsymbol{b}+\left(\frac{\boldsymbol{a}}{\boldsymbol{2}}\right)^2=\left(\frac{\boldsymbol{a}}{\boldsymbol{2}}+\boldsymbol{b}\right)^2$

The example of the cause:

➢ $a=10,\ \frac{a}{2}=5,\ b=3 \ \Rightarrow \ 10=5+5 \ \Rightarrow \ (10+3).3+25=(5+3)^2 \ \Rightarrow$ $13.3+25=8^2 \ \Rightarrow \ 39+25=64$

The application of the cause on the equation:

➢ $x^2+10x=24 \ \Rightarrow \ (10+2).2+\left(\frac{10}{2}\right)^2=(5+2)^2 \ \Rightarrow 24+25=49$

- **The cause of the second mixed equation type**

➢ $\forall\, \boldsymbol{a},\frac{\boldsymbol{a}}{\boldsymbol{2}},\boldsymbol{b},\boldsymbol{c} \in \boldsymbol{R}\backslash\{\boldsymbol{0}\} \Rightarrow \boldsymbol{a}=\frac{\boldsymbol{a}}{\boldsymbol{2}}+\frac{\boldsymbol{a}}{\boldsymbol{2}}$ ***and*** $\boldsymbol{a=b+c} \Rightarrow \left(\boldsymbol{b}-\frac{\boldsymbol{a}}{\boldsymbol{2}}\right)^2+\boldsymbol{c}.\boldsymbol{b}=$ $\left(\frac{\boldsymbol{a}}{\boldsymbol{2}}\right)^2$ ***or*** $\left(\frac{\boldsymbol{a}}{\boldsymbol{2}}-\boldsymbol{c}\right)^2+\boldsymbol{c}.\boldsymbol{b}=\left(\frac{\boldsymbol{a}}{\boldsymbol{2}}\right)^2$

Example of the cause:

- $a = 10,\ \frac{a}{2} = 5,\ b = 7\ and\ \ c = 3\ \Rightarrow\ (7-5)^2 + 7.3 = \left(\frac{10}{2}\right)^2$

The application of the cause on the equation:

- $x^2 + 16 = 10x\ \Rightarrow\ \ 10 = 2 + 8\ and\ 16 = 2.8\ \Rightarrow\ (8-5)^2 + 8.2 = \left(\frac{10}{2}\right)^2$

- **The cause of the third mixed equation type**

  - $\forall\, \boldsymbol{a}, \boldsymbol{b}\ \in \boldsymbol{R}\backslash\{\boldsymbol{0}\}\ \ \boldsymbol{and}\ \ \boldsymbol{a} \geq \left(\frac{\boldsymbol{b}}{2}\right)^2\ \ \Rightarrow\ \ \sqrt{\boldsymbol{a}} - \sqrt{\boldsymbol{a} - \boldsymbol{b}\sqrt{\boldsymbol{a}} + \left(\frac{\boldsymbol{b}}{2}\right)^2} = \frac{\boldsymbol{b}}{2}$

Example of the cause

- $a = 36\ \ and\ \ b = 4\ \ and\ \ 36 > \left(\frac{4}{2}\right)^2\ \Rightarrow\ \sqrt{36} - \sqrt{36 - 4\sqrt{36} + \left(\frac{4}{2}\right)^2} = \frac{4}{2}\ \ \Rightarrow$

  $6 - \sqrt{12 + 4} = 2\ \Rightarrow\ 6 - 4 = 2$

The application of cause on the equation:

- $x^2 = 10x + 24\ \ \Rightarrow\ \ x^2 - 10x = 24\ \ \Rightarrow\ \sqrt{x^2} - \sqrt{x^2 - 10x + \left(\frac{10}{2}\right)^2} = \frac{10}{2}\ \Rightarrow$

  $x = \sqrt{\boldsymbol{x^2} - \boldsymbol{10x} + \left(\frac{10}{2}\right)^2} + \frac{10}{2}\ \ \Rightarrow\ \sqrt{\boldsymbol{24} + \left(\frac{10}{2}\right)^2} + \frac{10}{2} = 12 = x$

7. Formulas that Give the Perfect Square in Mixed Equations[91]

- **Formulas to find the perfect square in the first mixed equation**

  - $\forall\, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\ \in \boldsymbol{R}\backslash\{\boldsymbol{0}\}\ \boldsymbol{and}\ \ \boldsymbol{ax^2} + \boldsymbol{bx} = \boldsymbol{c}\ \ \boldsymbol{and}\ \ \boldsymbol{c} < \frac{\boldsymbol{b^2}}{\boldsymbol{4}}\ \ \Rightarrow\ \ \boldsymbol{x^2} = \frac{\boldsymbol{b^2}}{\boldsymbol{2}} + \boldsymbol{c} -$

    $\sqrt{\boldsymbol{b^2}.\boldsymbol{c} + \left(\frac{\boldsymbol{b^2}}{\boldsymbol{2}}\right)^2}\ \ \boldsymbol{or}\ \ \boldsymbol{x^2} = \boldsymbol{c} + \frac{\boldsymbol{b^2}}{\boldsymbol{2}} - \sqrt{\left(\boldsymbol{c} + \frac{\boldsymbol{b^2}}{\boldsymbol{2}}\right)^2 - \boldsymbol{c^2}}\ \ \ \boldsymbol{or}\ \ \boldsymbol{x} =$

    $\frac{\sqrt{\boldsymbol{4c}+\boldsymbol{b^2}}-\boldsymbol{b}}{\boldsymbol{2}}\ \ \ \boldsymbol{and}\ \ \boldsymbol{x^2} = \frac{\left(\sqrt{\boldsymbol{4c}+\boldsymbol{b^2}}-\boldsymbol{b}\right)^2}{\boldsymbol{4}}$

- **Formulas to find the perfect square in the second mixed equation**

91 Ibid., 44b-46b.

➢ $\forall\, \boldsymbol{a, b, c} \in \boldsymbol{R}\backslash\{\boldsymbol{0}\}$ ***and*** $\boldsymbol{ax^2 + c = bx}$ ***and*** $\boldsymbol{c < \frac{b^2}{4}} \Rightarrow \boldsymbol{x_1^2 = \frac{b^2}{2} - \sqrt{\left(\frac{b^2}{2}\right)^2 - b^2.c} - c}$ ***and*** $\boldsymbol{x_2^2 = \frac{b^2}{2} + \sqrt{\left(\frac{b^2}{2}\right)^2 - b^2.c} - c}$ ***or*** $\boldsymbol{x_1 = \frac{b-\sqrt{b^2-4c}}{2}}$ ***and*** $\boldsymbol{x_1^2 = \frac{\left(b-\sqrt{b^2-4c}\right)^2}{4}}$ ***and*** $\boldsymbol{x_2 = \frac{b+\sqrt{b^2-4c}}{2}}$ ***and*** $\boldsymbol{x_2^2 = \frac{\left(b+\sqrt{b^2-4c}\right)^2}{4}}$ ***or*** $\boldsymbol{x_1^2 = \frac{b^2}{2} - c - \sqrt{\left(\frac{b^2}{2} - c\right)^2 - c^2}}$ ***and*** $\boldsymbol{x_2^2 = \left(\frac{b^2}{2} - c\right) + \sqrt{\left(\frac{b^2}{2} - c\right)^2 - c^2}}$

- **Formulas to find the perfect square in the third mixed equation**

➢ $\forall\, \boldsymbol{a, b, c} \in \boldsymbol{R}\backslash\{\boldsymbol{0}\}$ ***and*** $\boldsymbol{ax^2 = bx + c}$ ***and*** $\boldsymbol{c < \frac{b^2}{4}} \Rightarrow \boldsymbol{x^2 = \sqrt{b^2.c + \left(\frac{b^2}{2}\right)^2} + c + \frac{b^2}{2}}$ *or* $\boldsymbol{x^2 = \sqrt{\left(\frac{b^2+2c}{2}\right)^2 - c^2} + \frac{b^2+2c}{2}}$ ***or*** $\boldsymbol{x = \frac{\sqrt{4c+b^2}+b}{2}}$ ***and*** $\boldsymbol{x^2 = \frac{\left(\sqrt{4c+b^2}+b\right)^2}{4}}$

8. Formulas to Convert Mixed Equations to Simple Equations[92]

- **The premise to convert the first mixed equation to first or third simple equation and its formula**

➢ $\forall$ ***a and b are consecutive positive integers and*** $\boldsymbol{b > a} \Rightarrow \boldsymbol{\left(\frac{b-a}{2}\right)^2 + a.b = \left(\frac{b+a}{2}\right)^2}$

➢ $\forall\, \boldsymbol{a, b, c} \in \boldsymbol{R}\backslash\{\boldsymbol{0}\}$ ***and*** $\boldsymbol{ax^2 + bx = c} \Rightarrow$

for the first simple equation $\boldsymbol{\sqrt{ax^2.c + \left(\frac{bx}{2}\right)^2} = \frac{ax^2+bx+ax^2}{2}}$ ***and for the third simple equation*** $\boldsymbol{\sqrt{ax^2.c + \left(\frac{bx}{2}\right)^2} + \frac{bx}{2} = c}$

92 Ibid., 46b-48b.

- **Formulas to convert the second mixed equation to first and third simple equation**

  ➢ $\forall\, a, b, c \in R \setminus \{0\}$ *and* $ax^2 + c = bx \Rightarrow$

  for the first simple $\sqrt{\left(\frac{bx}{2}\right)^2 - ax^2.c} + \frac{bx}{2} = ax^2$ *or* $\frac{bx}{2} - \sqrt{\left(\frac{bx}{2}\right)^2 - ax^2.c} = ax^2$ *for the third simple* $\sqrt{\left(\frac{bx}{2}\right)^2 - ax^2.c} + \frac{bx}{2} = c$ *or* $\frac{bx}{2} - \sqrt{\left(\frac{bx}{2}\right)^2 - ax^2.c} = c$

- **Formula to convert the third mixed equation to first or third simple equation**

  ➢ $\forall\, a, b, c \in R \setminus \{0\}$ *and* $ax^2 = bx + c \Rightarrow$

  for the first simple $\sqrt{ax^2.c + \left(\frac{bx}{2}\right)^2} + \frac{bx}{2} = ax^2$ *and for the third simple* $\sqrt{ax^2.c + \left(\frac{bx}{2}\right)^2} - \frac{bx}{2} = c$

9. Converting the Equation by Means of Completion and Deletion[93]

- **First method/formula:**

  ➢ $\forall\, a, b, c \in R \setminus \{0\} \Rightarrow$ *in the equations* $ax^2 + c = bx$, $ax^2 = bx + c$ *and* $ax^2 + bx = c$, $a \neq 1 \Rightarrow ax^2.\frac{1}{a} + c.\frac{1}{a} = bx.\frac{1}{a}$, $ax^2.\frac{1}{a} = bx.\frac{1}{a} + c.\frac{1}{a}$ *and makes* $ax^2.\frac{1}{a} + bx.\frac{1}{a} = c.\frac{1}{a}$.

- **Second method/formula:**

  ➢ $a < 1 \rightarrow \frac{1-a}{a} \Rightarrow ax^2 + c + \left(\frac{1-a}{a}\right).(ax^2 + c) = bx + bx.\left(\frac{1-a}{a}\right)$ *and* $ax^2 + ax^2.\left(\frac{1-a}{a}\right) = bx + c + \left(\frac{1-a}{a}\right).(bx + c)$ *and makes* $ax^2 + bx + \left(\frac{1-a}{a}\right).(ax^2 +$

93 Ibid., 48b-51b.

$\boldsymbol{bx) = c + c.\left(\frac{1-a}{a}\right) \ . \ a > 1 \ \Rightarrow \ \frac{a-1}{a} \ \Rightarrow \ ax^2 + c - \left[\frac{a-1}{a}.(ax^2 + c)\right] = bx -}$

$\boldsymbol{bx.\left(\frac{a-1}{a}\right) \ and \ ax^2 - ax^2.\left(\frac{a-1}{a}\right) = bx + c - \left[\frac{a-1}{a}.(bx +\right.}$

$\boldsymbol{\left.c)\right] \ and \ makes \ ax^2 + bx - \left[\frac{a-1}{a}.(ax^2 + bx)\right] = c - c.\left(\frac{a-1}{a}\right).}$

- **Third method/formula:**

➢ $\boldsymbol{a \neq 1 \ \Rightarrow \ \frac{ax^2}{a} + \frac{c}{a} = \frac{bx}{a}, \quad \frac{ax^2}{a} = \frac{bx}{a} + \frac{c}{a} \quad and \ makes \ \frac{ax^2}{a} + \frac{bx}{a} = \frac{c}{a}}$.

10. Formulas to Solve the Equation without Completion and Deletion[94]

- $\boldsymbol{\forall \ a, b, c \ \in R\backslash\{0\} \ and \ a \neq 1 \ \Rightarrow \ ax^2 + bx = c \ \Rightarrow}$ **makes** $\boldsymbol{x = \frac{\sqrt{a.c + \left(\frac{b}{2}\right)^2} - \frac{b}{2}}{a}}$.

➢ $2x^2 + \frac{x^2}{2} + 10x = 150 \Rightarrow x = \frac{\sqrt{\left(2+\frac{1}{2}\right).150 + \left(\frac{10}{2}\right)^2} - \frac{10}{2}}{2+\frac{1}{2}} = \frac{\sqrt{375+25}-5}{\frac{5}{2}} = \frac{15}{\frac{5}{2}} = 6 \Rightarrow x = 6$

- $\boldsymbol{\forall \ a, b, c \ \in R\backslash\{0\} \ and \ a \neq 1 \ \Rightarrow \ ax^2 + c = bx \ \Rightarrow \ makes \ x_1 =}$

$\boldsymbol{\frac{\sqrt{\left(\frac{b}{2}\right)^2 - a.c} + \frac{b}{2}}{a}, \quad x_2 = \frac{\frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 - a.c}}{a}}$

➢ $x^2 + \frac{x^2}{3} + 12 = 10x \ \Rightarrow \ x_1 = \frac{\sqrt{\left(\frac{10}{2}\right)^2 - \left(1+\frac{1}{3}\right).12} + \frac{10}{2}}{1+\frac{1}{3}} = \frac{\sqrt{25-16}+5}{\frac{4}{3}} = \frac{8}{\frac{4}{3}} = \mathbf{6} \ \ and \ \ x_2 =$

$\frac{\frac{10}{2} - \sqrt{\left(\frac{10}{2}\right)^2 - \left(1+\frac{1}{3}\right).12}}{1+\frac{1}{3}} = \frac{5-\sqrt{25-16}}{\frac{4}{3}} = \frac{5-3}{\frac{4}{3}} = \mathbf{1 + \frac{1}{2}}$

- $\boldsymbol{\forall \ a, b, c \ \in R\backslash\{0\} \ and \ a \neq 1 \ \Rightarrow \ ax^2 = c + bx \ \Rightarrow \ makes \ x = \frac{\sqrt{a.c + \left(\frac{b}{2}\right)^2} + \frac{b}{2}}{a}}$

➢ $\frac{8x^2}{9} + \left(\frac{1}{2}.\frac{1}{9}\right)x^2 = 4x + 10 \ \Rightarrow \ x = \frac{\sqrt{\left(\frac{8}{9}+\frac{1}{2}\frac{1}{9}\right).10 + \left(\frac{4}{2}\right)^2} + \frac{4}{2}}{\frac{8}{9}+\frac{1}{2}\frac{1}{9}} = \frac{\sqrt{9+\frac{4}{9}+4}+2}{\frac{8}{9}+\frac{1}{2}\frac{1}{9}} = \frac{5+\frac{2}{3}}{\frac{8}{9}+\frac{1}{2}\frac{1}{9}} = \mathbf{6}$

11. Formulas to Find the Perfect Square Without Completion and Deletion[95]

94 Ibid., 51b-53a.

- $\forall\, \boldsymbol{a,b,c} \in \boldsymbol{R}\backslash\{\boldsymbol{0}\}$ ***and*** $\boldsymbol{a \neq 1}$ ***and*** $\boldsymbol{ax^2 + bx = c \Rightarrow x^2 = \dfrac{a.c+\frac{b^2}{2}-\sqrt{\left(a.c+\frac{b^2}{2}\right)^2-(a.c)^2}}{a^2}}$

$$➢\ 2x^2 + \frac{x^2}{2} + 10x = 150 \Rightarrow x^2 = \frac{\left(2+\frac{1}{2}\right).150+\frac{10^2}{2}-\sqrt{\left(\left(2+\frac{1}{2}\right).150+\frac{10^2}{2}\right)^2-\left(\left(2+\frac{1}{2}\right).150\right)^2}}{\left(2+\frac{1}{2}\right)^2} =$$

$$\frac{375+50-\sqrt{180625-140625}}{6+\frac{1}{4}} = \frac{425-\sqrt{40000}}{6+\frac{1}{4}} = \frac{425-200}{6+\frac{1}{4}} = \frac{225}{6+\frac{1}{4}} = \mathbf{36}$$

- $\forall\, \boldsymbol{a,b,c} \in \boldsymbol{R}\backslash\{\boldsymbol{0}\}$ ***and*** $\boldsymbol{a \neq 1}$ ***and*** $\boldsymbol{ax^2 + c = bx \Rightarrow {x_1}^2 =}$

$$\boldsymbol{\frac{\sqrt{\left(\frac{b^2-2c.a}{2}\right)^2-a^2.c^2}+\frac{b^2-2c.a}{2}}{a^2}}\quad \textbf{\textit{and makes}}\quad \boldsymbol{{x_2}^2 = \frac{\frac{b^2-2c.a}{2}-\sqrt{\left(\frac{b^2-2c.a}{2}\right)^2-a^2.c^2}}{a^2}}$$

$$➢\ \frac{5x^2}{6} + x^2\left(\frac{1}{2}.\frac{1}{6}\right) + 15 = 8x \quad\rightarrow\quad {x_1}^2 = \frac{\sqrt{\left(\frac{8^2-2.15.\left(\frac{5}{6}+\frac{1}{2}.\frac{1}{6}\right)}{2}\right)^2-\left(\frac{5}{6}+\frac{1}{2}.\frac{1}{6}\right)^2.15^2}+\frac{8^2-2.15.\left(\frac{5}{6}+\frac{1}{2}.\frac{1}{6}\right)}{2}}{\left(\frac{5}{6}+\frac{1}{2}.\frac{1}{6}\right)^2} =$$

$$\frac{\sqrt{\left(\frac{64-\left(27+\frac{1}{2}\right)}{2}\right)^2-\left(\frac{5}{6}+\frac{1}{2}.\frac{1}{8}.\frac{1}{9}\right).225}+\frac{64-\left(27+\frac{1}{2}\right)}{2}}{\frac{5}{6}+\frac{1}{2}.\frac{1}{8}.\frac{1}{9}} = \frac{\sqrt{\left(18+\frac{1}{4}\right)^2-\left(189+\frac{1}{2}.\frac{1}{8}\right)}+18+\frac{1}{4}}{\frac{5}{6}+\frac{1}{2}.\frac{1}{8}.\frac{1}{9}} =$$

$$\frac{\sqrt{333+\frac{1}{2}.\frac{1}{8}-\left(189+\frac{1}{2}.\frac{1}{8}\right)}+18+\frac{1}{4}}{\frac{5}{6}+\frac{1}{2}.\frac{1}{8}.\frac{1}{9}} = \frac{\sqrt{144}+18+\frac{1}{4}}{\left(\frac{2}{3}+\frac{1}{4}\right)^2} = \frac{12+18+\frac{1}{4}}{\left(\frac{2}{3}+\frac{1}{4}\right)^2} = \frac{30+\frac{1}{4}}{\left(\frac{2}{3}+\frac{1}{4}\right)^2} = 36 \quad\rightarrow\quad \boldsymbol{{x_1}^2} =$$

$$\mathbf{36} \quad \textit{and} \quad {x_2}^2 = \frac{\frac{8^2-2.15.\left(\frac{5}{6}+\frac{1}{2}.\frac{1}{6}\right)}{2}-\sqrt{\left(\frac{8^2-2.15.\left(\frac{5}{6}+\frac{1}{2}.\frac{1}{6}\right)}{2}\right)^2-\left(\frac{5}{6}+\frac{1}{2}.\frac{1}{6}\right)^2.15^2}}{\left(\frac{5}{6}+\frac{1}{2}.\frac{1}{6}\right)^2} =$$

$$\frac{\frac{64-\left(27+\frac{1}{2}\right)}{2}-\sqrt{\left(\frac{64-\left(27+\frac{1}{2}\right)}{2}\right)^2-\left(\frac{5}{6}+\frac{1}{2}.\frac{1}{8}.\frac{1}{9}\right).225}}{\frac{5}{6}+\frac{1}{2}.\frac{1}{8}.\frac{1}{9}} = \frac{18+\frac{1}{4}-\sqrt{\left(18+\frac{1}{4}\right)^2-\left(189+\frac{1}{2}.\frac{1}{8}\right)}}{\frac{5}{6}+\frac{1}{2}.\frac{1}{8}.\frac{1}{9}} =$$

$$\frac{18+\frac{1}{4}-\sqrt{333+\frac{1}{2}.\frac{1}{8}-\left(189+\frac{1}{2}.\frac{1}{8}\right)}}{\frac{5}{6}+\frac{1}{2}.\frac{1}{8}.\frac{1}{9}} = \frac{18+\frac{1}{4}-\sqrt{144}}{\left(\frac{2}{3}+\frac{1}{4}\right)^2} = \frac{18+\frac{1}{4}-12}{\left(\frac{2}{3}+\frac{1}{4}\right)^2} = \frac{6+\frac{1}{4}}{\left(\frac{2}{3}+\frac{1}{4}\right)^2} = \frac{900}{121} \quad\rightarrow\quad \boldsymbol{{x_1}^2 = \frac{900}{121}}$$

95 Ibid., 53a-54b.

- $\forall\, \boldsymbol{a, b, c} \in \boldsymbol{R}\backslash\{\boldsymbol{0}\}$ ***and*** $\boldsymbol{a \neq 1}$ ***and*** $\boldsymbol{ax^2 = c + bx} \Rightarrow \boldsymbol{x^2} = \dfrac{\frac{2c.a+b^2}{2} + \sqrt{\left(\frac{2c.a+b^2}{2}\right)^2 - a^2.c^2}}{a^2}$

➢ $2x^2 + \frac{2x^2}{3} = 10x + 36 \Rightarrow x^2 = \dfrac{\frac{2.36.\left(2+\frac{2}{3}\right)+10^2}{2} + \sqrt{\left(\frac{2.36.\left(2+\frac{2}{3}\right)+10^2}{2}\right)^2 - \left(2+\frac{2}{3}\right)^2.36^2}}{\left(2+\frac{2}{3}\right)^2} =$

$\dfrac{\frac{192+100}{2} + \sqrt{\left(\frac{192+100}{2}\right)^2 - \left(7+\frac{1}{9}\right).1296}}{\left(2+\frac{2}{3}\right)^2} = \dfrac{146+\sqrt{146^2-9216}}{\left(2+\frac{2}{3}\right)^2} = \dfrac{146+\sqrt{21316-9216}}{\left(2+\frac{2}{3}\right)^2} = \dfrac{146+\sqrt{12100}}{\left(2+\frac{2}{3}\right)^2} =$

$\dfrac{146+110}{\left(2+\frac{2}{3}\right)^2} = \dfrac{256}{\left(2+\frac{2}{3}\right)^2} = \mathbf{36}$

12. Examples of Higher-Degree Equations concerning the Indetermination of the Number of Equations[96]

➢ $x^4 + 24 = 10x^2 \rightarrow assume\ x^2 = x \rightarrow x^2 + 24 = 10x \rightarrow x_1 = 6\ and\ x_2 = 4 \rightarrow for\ assume\ x^2 = x,\ {x_1}^2 = 6\ and\ {x_2}^2 = 4,\ x_1 = \sqrt{6}\ and\ x_2 = 2,\ {x_1}^4 = 36\ and\ {x_2}^4 = 16$

➢ $x^7 = 28x + 4x^4 + \frac{x^4}{2} \rightarrow exponents\ 7, 4, 1\ that\ is\ consecutive\ by\ 3\ makes \rightarrow$

$x^2 = 28 + 4x + \frac{x}{2} \rightarrow x =$

$8\ and\ since\ it\ is\ consecutive\ by\ 3\ it\ in\ fact\ makes\ x^3 = 8\ and\ x = 2$

➢ $a + b = 10\ and\ b\sqrt{a} = 12 \rightarrow a = x^2\ and\ b = 10 - x^2 \rightarrow makes\ x(10 - x^2) = 12 \rightarrow 10x - x^3 = 12 \rightarrow 10x = 12 + x^3 \rightarrow 10x.x = x(12 + x^3) \rightarrow$

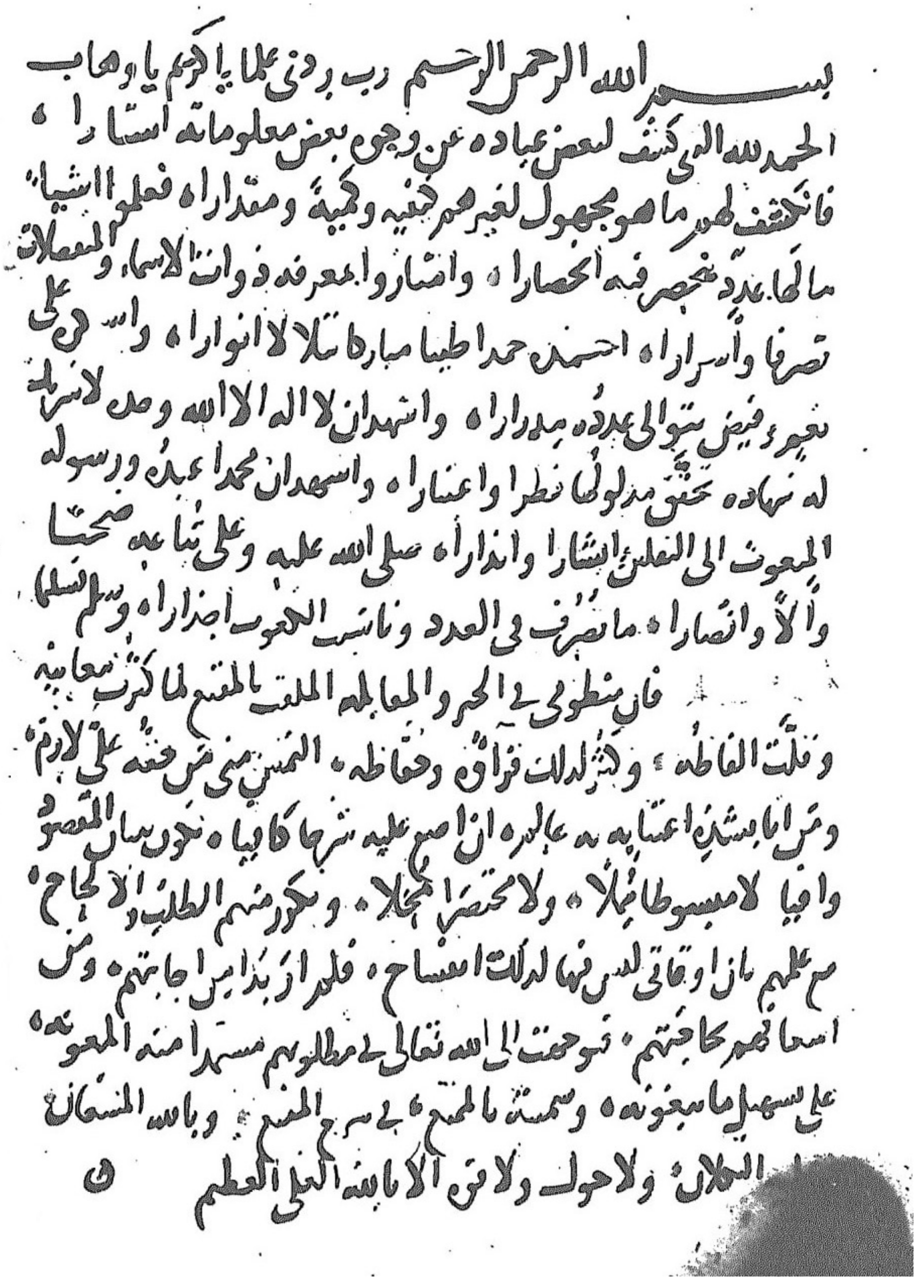
$10x^2 = x^4 + 12x \rightarrow 10x^2 - 12x = x^4 \rightarrow \sqrt{10x^2 - 12x} = \sqrt{x^4} \rightarrow$

$\sqrt{10x^2 - 12x} = x^2 \rightarrow x^2 = 2x\ assume \rightarrow \sqrt{10x^2 - 12x} = 2x \rightarrow 10x^2 - 12x = 4x^2 \rightarrow x = 2, a = 4, b = 6$

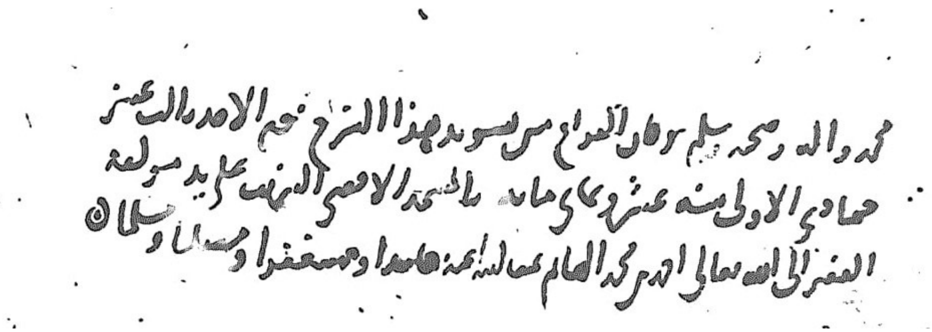
96 Ibid., 57a-60a.

- $\boldsymbol{x^4 + 2x^3 = x + 30} \rightarrow (x^2 + x)^2 = x^4 + 2x^3 + x^2 \ \rightarrow \ x^4 + 2x^3 + x^2 = x^2 + x + 30 \ \ and \ \ (x^2 + x)^2 = x^2 + x + 30 \ \ \rightarrow \ \ x^2 + x = y \ \ assume \ \rightarrow \ y^2 = y + 30 \ \ \rightarrow \ \ y = 6 \rightarrow \ x^2 + x = 6 \ \ \rightarrow \ \ x = 2, \ x^2 = 4, \ x^3 = 8, \ x^4 = 16 \ \rightarrow \ x^4 + 2x^3 = 16 + 16 = 32$

Appendix 3: The First and Last Folios of the Autograph Copy of al-Mumti' at Chester Beatty 3881:


بسم الله الرحمن الرحيم رب زدني علما يا اكرم يا وهاب
الحمد لله الذي كشف لبعض عباده عن وجوه بعض معلوماته استارا
فانكشف لهم ما هو مجهول لغيرهم جنسه وكميته ومقداره فعلموا الاشياء
ما لها عدد ينحصر فيه انحصاره واشاروا بمعرفة ذوات الاسماء والمعلات
تصرفها واسراره احمده حمدا طيبا مباركا تتلألأ انواره واشكره على
نعمه فيض يتوالى بعدده مدراره واشهد ان لا اله الا الله وحده لا شريك
له شهادة تحقق مدلولها نظرا واعتباره واشهد ان محمدا عبده ورسوله
المبعوث الى الثقلين ابشارا وانذاره صلى الله عليه وعلى تابعيه وصحبه
والآل وانصاره ما تصرف في العدد ونابت الكعوب اجذاره وسلم تسليما
فان منظومتي في الجبر والمقابلة الملقب بالمقنع لما كثرت معانيه
وقلت الفاظه وكثرت لذلك قراؤه وحفاظه التمس مني من حفظه على الارض
ومن اما بشيء اعتنى به ان اضع عليه شرحا كافيا ونحو بيان المقصود
وافيا لا مبسوطا مملا ولا مختصرا مخلا وتكرر منهم الطلب والالحاح
مع علمهم باني … فيها لذلك انفساح فلما ازداد ابن حاجتهم ومن
اسعافهم كاحبتهم توجهت الى الله تعالى في مطلوبهم مستمدا منه المعونة
على تسهيل ما يبغونه وسميته بالممتع في شرح المقنع وبالله المستعان
التكلان ولا حول ولا قوة الا بالله العلي العظيم

واعمل على الخامس يخرج المطلوب وان شئت فاضرب الامس في السدس في العشر
الاشيا واطرح من الكامل الشيء واضرب الباقي في الشيء وعادل بها الكامل مع
العشر الاشيا يخرج ايضا للضرب الخامس وان شئت فسم عشرة الاشيا على شيء
وسمي الخارج مجهولا من المجهولات باي اسم شئت مثلا دينار فهي ضرب في سدس الشيء
حرج عشرة الاشيا ويكون لذلك الخارج من قسمة الشيء على العشر الاشيا دينار
وسدسها الا دينارا فاضرب في المقسوم عليه وهو عشر الاشيا واعمل الخارج من
ضرب الشيء في الدينار عشرة الاشيا لان الخارج من القسمة اذا ضرب في المقسوم عليه
يخرج المقسوم يكون الخارج احدا وثلاثين وثلثين الا ثلاثة اشيا وسدس شيء والا عشر
دنانير تعادل به لذلك الشيء المقسوم واجبر حتى يكون احد وثلثون وثلثان يعدل
اربعة اشيا وسدسها وعشر دنانير فاطرح اربعة الاشيا والسدس من الجانبين يبقى
ستة احد وثلثون وثلثان الا اربعة اشيا وسدس شيء يعدل عشرة دنانير فالدينار
الواحد يعدل ثلاثة وسدسا الا اربع وسدس عشر ذا فرضنا الخارج من ضرب الدينار
في الشيء عشرة الاشيا فاقم مقام الدينار ما عادله واضربه في الشيء فيخرج ثلاثة اشيا
وسدس شيء الا اربع وسدس مال وذلك يعدل عشرة الاشيا فاجبر وقابل واعمل
ستحق فالمطلوب هذه الطرق ونذكر ما فيها من وجوه التحيل على الوصول الى المطلوب
ونشر عليها ما يرد من اشباهها وبالله التوفيق وفي هذا الذي اوردناه كفاية ومقنع
لمن اراد الله بالتوفيق ولله الحمد اولا واخرا وظاهرا وباطنا وصلى الله على سيدنا



محمد وآله وصحبه وسلم وكان الفراغ من تسويد هذا الشرح ضحى الأحد ثالث عشر
جمادى الأولى سنة عشر وثماني مائة بالمسجد الأقصى الشريف على يد مؤلفه
الفقير إلى الله تعالى أحمد بن محمد الهائم عفا الله عنه حامدا ومستغفرا ومصليا ومسلما